Partial differential Equation UNIT-II

Definition: An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a P.D.E.

For examples of partial diff. Equations

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy \quad \text{---(1)} \qquad \left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right) \quad \text{---(2)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz \quad \text{---(3)} \qquad \frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2} \quad \text{---(4)}$$

Order of a partial diff. Equation: The order of a P.D.E. is defined as the order of the highest partial derivative occuring in the partial differential Equation.

Equation (1) and (3) are of the first order

(4) is of the second order (2) is of the 3rd order

Degree of a P.D.E is the degree of the highest order derivative which occurs in it after the equation has been rationalised i.e. made free from radicals and fractions

Equation (1), (3) are of first degree

Equation (4) are of 2nd degree

Notations → we adopt the following notations throughout the study of partial differential Equations

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

Some times the partial differentiations are also denoted by making use of suffixes.

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y} \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

-- and so, on

Formation of a Partial differential Equation

Rule-I Derivation of a Partial differential Equation by the elimination of arbitrary Constants.

Consider an equation $f(x, y, z, a, b) = 0$ --- (1)

where a and b are arbitrary constant.

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{i.e} \quad \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \quad \text{--- (2)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \quad \text{i.e} \quad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \quad \text{--- (3)}$$

Eliminating two constants a and b from (1), (2) and (3) we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0 \quad \text{--- (4)}$$

Which is a partial differential equation of first order.

Examples →.

Ex-1 Find a partial differential Equation from $z = ax + by + a^2 + b^2$ where a and b are arbitrary constant.

Sol. Given $z = ax + by + a^2 + b^2$ —— (1)

D. (1) P. w.r.to x and y we get

$$p = \frac{\partial z}{\partial x} = a, \quad \text{and} \quad \frac{\partial z}{\partial y} = b = q$$

above values put in (1) we obtain

$$z = px + qy + p^2 + q^2.$$

Ex-2 Form the partial differential Equation by eliminating h and k from the Equation $(x-h)^2 + (y-k)^2 + z^2 = d^2$

Sol. Given $(x-h)^2 + (y-k)^2 + z^2 = d^2$ --- (1)

Differentiating (1) partially w.r.to x and y, we get

$$2(x-h)^2 + 2z\left(\frac{\partial z}{\partial x}\right) = 0 \Rightarrow (x-h) = -zp \quad \text{--- (2)}$$

$$2(y-k)^2 + 2z\left(\frac{\partial z}{\partial y}\right) = 0 \Rightarrow (y-k) = -zq \quad \text{--- (3)}$$

Substituting the values of $(x-h) = -zp$, and $y-k = -zq$ in (1)

$$z^2(p^2 + q^2 + 1) = d^2$$

Ex-3 Find the differential Equation of all spheres of radius d, having centres in the xy plane.

Sol. The equation of any sphere of radius d having centre $(h, k, 0)$ in the xy plane is given by $(x-h)^2+(y-k)^2+(z-0)^2=d^2$

or $(x-h)^2+(y-k)^2+z^2=d^2$ ---(1)

Now proceed exactly in the same way as Ex-2

Exercise-

Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding partial differential Equations

Ex-(a) $\log(az-1)=x+ay+b$ (a and b)

(b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (a, b and c)

(c) $2z=(ax+y)^2+b$ (a and b)

Rule-II Derivation of partial diff. Eqn by the elimination of arbitrary function ϕ from the equation $\phi(u,v)=0$ where u and v are functions of x, y and z

Ex Form a partial diff. Eqn by eliminating the arbitrary function ϕ from $\phi(x+y+z, x^2+y^2+z^2)=0$

Sol $\phi(x+y+z, x^2+y^2-z^2)=0$ ---(1)

Let $u=x+y+z$ and $v=x^2+y^2+z^2$ — (2)

Then (1) becomes $\phi(u,v)=0$ --- (3)

D. (3) w.r. to 'x' partially we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \quad \text{—(4)}$$

From(2) $\frac{\partial u}{\partial x}=1$, $\frac{\partial u}{\partial z}=1$, $\frac{\partial v}{\partial x}=2x$, $\frac{\partial v}{\partial z}=-2z$, $\frac{\partial u}{\partial y}=1$, $\frac{\partial v}{\partial y}=2y$ —(5)

From (4) and (5) $\left(\frac{\partial \phi}{\partial u}\right)(1+p) + 2\left(\frac{\partial \phi}{\partial v}\right)(x-pz) = 0$

or $\left(\frac{\partial \phi}{\partial u}\right)\left(\frac{\partial \phi}{\partial v}\right) = -\frac{2(x-pz)}{1+p}$ --- (6)

Again D. (3) w.r.t 'y' partially we get

$$\left(\frac{\partial \phi}{\partial u}\right)\left(\frac{\partial \phi}{\partial v}\right) = -\frac{2(y-qz)}{1+q} \quad \text{--- (7)}$$

Eliminating ϕ from (6) and (7) we obtain

$$\frac{(x-pz)}{1+p} = \frac{y-qz}{(1+q)} \quad \text{or} \quad (1+q)(x-pz) = (1+p)(y-qz)$$

or $(y+z)p - (x+z)q = x-y$

which is the desired P.D.E. of 1st order.

Exercise

(a) Form a P.D.E by eliminating the arbitrary function f from the equation $x+y+z = f(x^2+y^2+z^2)$

(b) Form a partial differential Equation by eliminating the function ϕ from $lx+my+nz = \phi(x^2+y^2+z^2)$

(c) Form a P.D.E by eliminating the arbitrary functions ϕ from $\phi(x^2+y^2+z^2,\ z^2-2xy) = 0$

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Linear P.D.E of order one

Lagrange's Equation → A quasi-linear partial differential Equation of order one is of the form

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

Such a partial diff. Eqn is known as Lagrange, equation

Working Rule for Solving $Pp + Qq = R$ by Lagrange's method

Step-1 Put the given linear Partial diff. Eqn of the 1st order in the standard form $Pp + Qq = R$ — ①

Step-2 Write down Lagrange' auxiliary eqn for ① namely

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{— ②}$$

Step-3 Solve ② Taking any two ratios ie I & II, I & III or II & IIIrd.

Let $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ be two independent Solutions of ②

Step-4 The general Solution (or integral) of ① is then written in one of the following three equivalent forms

$\phi(u,v) = 0$, $u = \phi(v)$ or $v = \phi(u)$

ϕ being an arbitrary function

<u>Ex</u> Solve $(y^2z/x)p + xzq = y^2$.

Sol Given $\frac{y^2z}{x}p + xzq = y^2$. — ①

The Lagrange's auxilary equations for ① are

$$\frac{dx}{y^2z/x} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \text{— ②}$$

Taking the first two fractions of ② we have $x^2z\,dx = y^2z\,dy$

$3x^2dx - 3y^2dy = 0$

Integrating $x^3 - y^3 = c_1$, c_1 being an arbitrary const. — ③

Next taking 1st and 3rd fractions of ② we get

$$xy^2 dx = y^2 dz \Rightarrow 2x dx - 2z dz = 0$$

Integrating $x^2 - z^2 = c_2$. — (4)

from ③ and ④ the required general integral is

$$\phi(x^3 - y^3, x^2 - z^2) = 0$$

ϕ being an arbitrary function.

Ex Solve $y^2 p - xyq = x(z - 2y)$

Sol Here Lagrange's auxilary eqn

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \quad \text{—①}$$

Taking 1st and II fractions of ①

$$2x dx + 2y dy = 0$$

$$x^2 + y^2 = c_1. \quad \text{—②}$$

Now taking IInd and IIIrd ans

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dz}{dy} = -\frac{z-2y}{y} \quad \text{or} \quad \frac{dz}{dy} + \frac{1}{y} z = 2 \quad \text{—③}$$

Which is linear in z & y $\quad I.F = e^{\int \frac{1}{y} dy} = e^{\log y} = y$

Hence Solution of ③ $\quad z \times y = \int 2y\, dy + c_2$

$$zy = y^2 + c_2$$

$$zy - y^2 = c_2$$

Hence $\phi(x^2 + y^2, zy - y^2) = 0$ is the desired Solution

Ex- Solve the following P.D. Equations

ⓐ $x^2 p + y^2 q = z^2$

ⓑ $yzp + 2xq = xy$

ⓒ $y^2 p^2 + x^2 q^2 = x^2 y^2 z^2$

ⓓ

Type-2

$Pp + Qq = R$ —①

Suppose that one integral of ① is known by using rule-I and also the another integral cannot be obtained by using Rule I. Then one integral known to us is used to find another integral.

Ex-1 Solve $p + 3q = 5z + \tan(y-3x)$.

Sol Given $p + 3q = 5z + \tan(y-3x)$ —①

The auxiliary (Lagranges) equations are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$$

Taking I & II $3dx = dy$

Integrating $y - 3x = c_1$ —②

Taking I & III using ② $\frac{dx}{1} = \frac{dz}{5z + \tan c_1}$

Integrating $x - \frac{1}{5}\log(5z + \tan c_1) = \frac{1}{5}c_2$

$5x - \log[5z + \tan(y-3x)] = c_2$ —③

From ② and ③

$$5x - \log(5z + \tan(y-3x)) = \phi(y-3x)$$

ϕ is an arbitrary constant

Ex-2 Solve $xyp + y^2q = zxy - 2x^2$.

Sol Given $xyp + y^2q = zxy - 2x^2$ —①

The Lagrange's subsidiary equations for ① are

$$\frac{dx}{xy} = \frac{dy}{yx} = \frac{dz}{zxy - 2x^2} \quad —②$$

Take I & II $\frac{dx}{x} = \frac{dy}{1} \Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = 0$

Integrating $\log x - \log y = \log c_1 \Rightarrow \boxed{\frac{x}{y} = c_1}$ —③

~~Type-2~~ From (3) $x = c_1 y$

P-8

From II & III rd $\dfrac{dy}{y^2} = \dfrac{dz}{c_1 z y^2 - 2c_1^2 y^2}$

or $c_1 dy - \dfrac{dz}{z - 2c_1^2}$ — (4)

Integrating (4) $c_1 y = \log(z - 2c_1^2) = c_2$.

$x - \log[z - 2(x^2/y^2)] = c_2$ — (5)

From (3) and (5), the required Solution

Ex Solve $py + qx = xyz^2(x^2 - y^2)$.

Sol Given $py + qx = xyz^2(x^2 - y^2)$ — (1)

The Lagranges Subsidiary Eqns are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)} \quad \text{— (2)}$$

Taking I & II $\boxed{x^2 - y^2 = c_1}$ — (3)

Taking II & III using (3) $\dfrac{dy}{\cancel{x}} = \dfrac{dz}{xyz^2 c_1}$

Integrating $y^2(x^2 - y^2) + \dfrac{2}{z} = c_2$

$\Rightarrow$ $y^2(x^2 - y^2) + \dfrac{2}{z} = \phi(x^2 - y^2)$, ϕ ~~being~~ is an arbitrary function

Exercise

1. Solve $p - 2q = 3x^2 \sin(y + 2x)$
2. Solve $xy^2 p - y^3 q + axz = 0$
3. Solve $z(p - q) = z^2 + (x + y)^2$

Type-3 (Multiplier Method) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ — ① &

we can write from well known principle of algebra each fraction in ①

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

we choose l, m, n in such a way that $lP + mQ + nR = 0$

Then $l\,dx + m\,dy + n\,dz = 0 \Rightarrow lx + my + nz = c_1$

$\Rightarrow$ which give one of the solution

Ex-1 Solve $(mz - ny)p + (nx - lz)q = ly - mx$

Sol. The Lagrange's auxiliary equations

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \quad —①$$

Choosing x, y, z as multipliers, each fraction of ①

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$

integrating $x^2 + y^2 + z^2 = c_1$ — ②

Again choosing l, m, n as multipliers, each fraction of ①

$$\frac{l\,dx + m\,dy + n\,dz}{0}$$

$\therefore$ $l\,dx + m\,dy + n\,dz = 0$

integrating $lx + my + nz = c_2$ — ③

From ② and ③ the required solution

$f(x^2 + y^2 + z^2,\ lx + my + nz) = 0$

Ex-2 Solve $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

Sol:- Here Lagrange's subsidiary equations for given by

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad —①$$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction of ①

$$= \frac{1/x\,dx + 1/y\,dy + \frac{1}{z}\,dz}{0} \Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$\Rightarrow \log x + \log y + \log z = \log c_1 \Rightarrow \boxed{xyz = c_1}$

Choosing $x, y, -1$ as multipliers, each fraction of ①

$$= \frac{x\,dx + y\,dy - dz}{0} \Rightarrow x\,dx + y\,dy - dz = 0$$

$$x^2 + y^2 - 2z = c_2$$

The required solution $\phi(x^2+y^2-2z, xyz) = 0$

Ex Solve $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$
If the solution of the above equation represents a sphere, what will be the co-ordinates of its centre.

Sol Here Lagrange's auxiliary equations

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} \quad \text{— ①}$$

Taking last two fractions of ① we have

$$2y\,dy - 2z\,dz - 2(z\,dy + y\,dz) = 0$$

integrating $y^2 - z^2 - 2yz = c_1$, c_1 being arbitrary const. ②

Choosing x, y, z as multipliers, each fraction of ①

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{0} \Rightarrow x\,dx + y\,dy + z\,dz = 0$$

$$x^2 + y^2 + z^2 = c_2 \quad \text{— ③}$$

From ② and ③, Solution is $\phi(y^2 - z^2 - 2yz, x^2 + y^2 + z^2) = 0$

Exercise

Ex-1 Solve $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$

Ex-2 Solve $(y + zx)p - (x + yz)q = x^2 - y^2$

Ex-3 Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

Ex Solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$

Non-Linear partial differential equations of 1st order

e-g $p^2+q^2=1$, $pq=z$, $x^2p^2+y^2q^2=z^2$.

Compatible System of 1st order Equations →

Consider first order partial differential equations

$$f(x,y,z,p,q)=0 \quad \text{---- (1)}$$

and

$$g(x,y,z,p,q)=0 \quad \text{---- (2)}$$

Equations (1) and (2) are known as compatible when every solution of one is also solution of the other.

Condition for (1) and (2) to be compatible →

Let J = Jacobian of f and g $\equiv \dfrac{\partial(f,g)}{\partial(p,q)} \neq 0$ —(3)

Then (1) and (2) can be solved to obtain the explicit expression for p and q given by

$$p=\phi(x,y,z) \ \& \ q=\psi(x,y,z) \quad \text{—(4)}$$

The condition that the pair of equations (1) and (2) should be compatible reduces then to the condition that the system of equations (4) should be completely integrable i.e the equation $dz = p\,dx + q\,dy$

or $\phi\,dx+\psi\,dy-dz=0$ using (4) —(5)

should be integrable* if

$$\phi\left(\frac{\partial\psi}{\partial z}-0\right)+\psi\left(0-\frac{\partial\phi}{\partial z}\right)+(-1)\left(\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x}\right)=0$$

Which is equivalent to $\dfrac{\partial\psi}{\partial x}+\phi\dfrac{\partial\psi}{\partial z}=\dfrac{\partial\phi}{\partial y}+\psi\dfrac{\partial\phi}{\partial z}$ (6)

Substituting from equation (4) in (1) and differentiating w.r.t 'x' and 'z' respectively we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{\partial\phi}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial\psi}{\partial x}=0 \quad \text{—(7)}$$

and

$$\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\frac{\partial\phi}{\partial z}+\frac{\partial f}{\partial q}\frac{\partial\psi}{\partial z}=0 \quad \text{—(8)}$$

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* $P\,dx+Q\,dy+R\,dz=0$ is integrable if $P\left(\dfrac{\partial Q}{\partial z}-\dfrac{\partial R}{\partial y}\right)+Q\left(\dfrac{\partial R}{\partial x}-\dfrac{\partial P}{\partial z}\right)+R\left(\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right)=0$

From (7) and (8) $\frac{\partial f}{\partial x} + \phi \frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\left(\frac{\partial \phi}{\partial x} + \phi \frac{\partial \phi}{\partial z}\right) + \frac{\partial f}{\partial q}\left(\frac{\partial \psi}{\partial x} + \phi \frac{\partial \psi}{\partial z}\right) = 0$ —(9)

Similarly (2) yields $\frac{\partial g}{\partial x} + \phi \frac{\partial g}{\partial z} + \frac{\partial g}{\partial p}\left(\frac{\partial \phi}{\partial x} + \phi \frac{\partial \phi}{\partial z}\right) + \frac{\partial g}{\partial q}\left(\frac{\partial \psi}{\partial x} + \phi \frac{\partial \psi}{\partial x}\right)$ —(10)

Solving (9) & (10)

$$\frac{\partial \psi}{\partial x} + \phi \frac{\partial \psi}{\partial z} = \frac{1}{J}\left[\frac{\partial (f,g)}{\partial (x,p)} + \phi \frac{\partial (f,g)}{\partial (z,p)}\right] \quad \text{—(11)}$$

Again substituting from equ (4) in (1) and differentiating w.r.t 'y' and 'z' and proceeding as before we obtain

$$\frac{\partial \phi}{\partial y} + \phi \frac{\partial \phi}{\partial z} = -\frac{1}{J}\left[\frac{\partial (f,g)}{\partial (y,q)} + \psi \frac{\partial (f,g)}{\partial (z,q)}\right] \quad \text{—(12)}$$

Substituting from equ's (11) and (12) in (1) and replacing ϕ, ψ by p, q respectively, we obtain

$$\frac{1}{J}\left[\frac{\partial (f,g)}{\partial (x,p)} + p \frac{\partial (f,g)}{\partial (z,p)}\right] = \frac{1}{J}\left[\frac{\partial (f,g)}{\partial (y,q)} + q \frac{\partial (f,g)}{\partial (z,q)}\right] \text{ or } [f,g] = 0 \quad \text{—(13)}$$

$$\boxed{[f,g] \equiv \frac{\partial (f,g)}{\partial (x,p)} + p \frac{\partial (f,g)}{\partial (z,p)} + \frac{\partial (f,g)}{\partial (y,q)} + \frac{\partial (f,g)}{\partial (z,q)}} \quad \text{—(14)}$$

Ex Show that the equations $xp = yq$ and $z(xp+yq) = 2xy$ are compatible and solve them

Sol Let

$$f(x,y,z,p,q) = xp - yq = 0 \quad \text{---(1)}$$
$$g(x,y,z,p,q) = z(xp+yq) - 2xy = 0 \quad \text{---(2)}$$

$$\frac{\partial (f,g)}{\partial (x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} p & x \\ zp-2y & xz \end{vmatrix} = 2xy$$

$$\frac{\partial (f,g)}{\partial (z,p)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ \partial g/\partial z & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp+yq & xz \end{vmatrix} = -x^2p - xyq$$

$$\frac{\partial (f,g)}{\partial (y,q)} = -2xy, \quad \frac{\partial (f,g)}{\partial (z,q)} = xyp + y^2q$$

$$[f,g] = \frac{\partial (f,g)}{\partial (x,p)} + p\frac{\partial (f,g)}{\partial (z,p)} + \frac{\partial (f,g)}{\partial (y,q)} + q\frac{\partial (f,g)}{\partial (z,q)} = 2xy - x^2p^2 - xyqp - 2xy + xypq + y^2q^2$$

$$= -xp(xp+yq) + yq(xp+yq) = 0 \quad \text{using (1)}$$

Hence (1) and (2) are compatible

Solving (1) and (2) for p, q, $p = y/z$, $q = x/z$ —(3)

$$dz = p\,dx + q\,dy = \frac{y}{z}dx + \frac{x}{z}dy$$

Integrating

$$z^2 = 2xy + c \quad (c \text{ is an arbitrary constant})$$

Ex-2 Show that the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ which is homogeneous in x, y, z

Sol Given that differential equation

$$f(x, y, z, p, q) = 0 \quad \text{— (1)}$$

is homogeneous in x, y, z, of degree n then by Euler's theorem on homogeneous functions, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

So that $\quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 0 \quad$ — (2) (from (1)).

We take $\quad g(x, y, z, p, q) = px + qy - z = 0 \quad$ —— (3)

Then using (3) we have

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ p & x \end{vmatrix} = x\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial p}$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ -1 & x \end{vmatrix} = x\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \partial f/\partial y & \partial f/\partial q \\ q & y \end{vmatrix} = y\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial q}$$

and $\quad \dfrac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial q \\ -1 & y \end{vmatrix} = y\dfrac{\partial f}{\partial z} + \dfrac{\partial f}{\partial q}$

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$$

$$= x\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial p} + p\left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\right) + y\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial q}$$

$$+ q\left(y\frac{\partial f}{\partial z} + \frac{\partial f}{\partial q}\right) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} \quad \text{using (3)}$$

$$= 0$$

$(z = px + qy)$ which is compatible with any differential equation $f(x,y,z,p,q) = 0$

Exercise

Ex-1 Show that $p^2 + q^2 = 1$ and $(p^2 + q^2)x = pz$ are compatible and solve them.

Ex-2 Show that $p = y(2ax + by)$, $q = x(ax + 2by)$ are compatible and solve them.

Charpit method (For Non linear diff. Equations).

Let the given partial differential equations of first order and higher degree in p and q be

$$f(x, y, z, p, q) = 0 \quad ----(1)$$

Therefore charpit auxiliary equations (1) may be re-written as

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0} \quad -(2)$$

or

$$\frac{dp}{\partial f/\partial x + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{df}{0}$$

Working Rule while using Charpit Method →

S-1 Transfer all terms of the given equation to L.H.S and denote the entire expression by f

S-2 Write down the charpit's auxiliary equations (2)

S-3 Find out $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial p$, $\partial f/\partial q$, $\partial f/\partial z$ and put these in charpit auxiliary equations

S-4 taking any fractions find out p and q put these values of p and q in $dz = p\,dx + q\,dy$ and integrating.

Ex Find complete integral of $z = px + qy + p^2 + q^2$.

Sol - Let $f \equiv z - px - qy - p^2 - q^2$.

$f_x = -p$, $f_y = -q$, $f_z = 1$, $f_p = -x - 2p$, and $f_q = -y - 2q$

using (2), (2) reduces to $\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q}$

$$\frac{dp}{0} = \frac{dq}{0}$$

$\Rightarrow$ $dp = 0$, $\quad dq = 0$

$\Rightarrow$ $\boxed{p = a}$ $\quad \boxed{q = b}$

put in (1), the required complete integral is

$$\boxed{z = ax + by + a^2 + b^2}$$

a, b, being arbitrary constants.

Ex-2 Find a complete integral of $p^2 - y^2 q = y^2 - x^2$

Sol Here given equation is $f \equiv p^2 - y^2 q - y^2 + x^2 = 0$ —(1)

Charpit's auxiliary equations

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{2x} = \frac{dq}{-2qy - 2y} = \frac{dz}{-p(2p) - q(-q^2)} = \frac{dx}{-2p} = \frac{dy}{y^2} \quad \text{using (1)}$$

Taking 1st and 4th fractions $p\,dp + x\,dx = 0$ So that $p^2 + x^2 = a^2$ --(3)

$\Rightarrow \boxed{p = (a^2 - x^2)^{1/2}}$

$q = \frac{a^2}{y^2} - 1$

$\therefore dz = p\,dx + q\,dy$

$= (a^2 - x^2)^{1/2} dx + (a^2 y^{-2} - 1)\,dy$

Integrating $z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{a^2}{y} - y + b$

Exercise

Ex-1 Find a complete integral of $z^2(p^2 z^2 + q^2) = 1$

Ex-2 Find a complete integral of $px + qy = pq$

Ex-3 Find the complete integral of the following equations

(a) $p = (z + qy)^2$

(b) $(p^2 + q^2)x = pz$

(c) $2xz - px^2 - 2qxy + pq = 0$

(d) $p^2 + q^2 - 2px - 2qy + 2xy = 0$

(e) $p^2 + q^2 - 2px - 2qy + 1 = 0$

(f) $2z + p^2 + qy + 2y^2 = 0$

Standard Forms – I

1. **Only p and q present** → $f(p,q)=0$

$f_x = f_y = f_z = 0$

Charpit equation $\frac{dp}{0} = \frac{dq}{0} \Rightarrow dp = 0 \Rightarrow \boxed{p=a}$ ~~dq=0~~ $(dq=0)$

& q is $q = \phi(a)$

$dz = a\,dx + \phi(a)\,dy$

$$\boxed{z = ax + \phi(a)y + b}$$

Rule-1 To Solve $f(p,q)=0$

Rule-2 Take $p=a$ in the given equation and find q in terms of a

Rule-3 put the values of p and q in $dz = p\,dx + q\,dy$

Ex-1

Solve $pq = 1$

Taking $p=a$, $\Rightarrow q = \frac{1}{a}$.

The equations $dz = a\,dx + \frac{1}{a}\,dy$

$z = ax + \frac{1}{a}y + b$ a & b being arbitrary

Ex-2 Solve $p^2+q^2=1$.

Sol. Taking $p=a$, $\Rightarrow q = \sqrt{1-a^2}$.

then $dz = a\,dx + \sqrt{1-a^2}\,dy$

Integrating $z = ax + \sqrt{1-a^2}\,y + b$

Ex Find the complete integral of $x^2p^2 + y^2q^2 = z$.

$$\Rightarrow \left(\frac{x}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = 1.$$

put $\frac{1}{x}dx = dX$ $\quad \frac{1}{y}dy = dY$ $\quad \left(\frac{1}{\sqrt{z}}\right)dz = dZ$

$\log x = X$ $\quad \log y = Y$ $\quad 2\sqrt{z} = Z$

using (2), (1) becomes $\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$, $P^2+Q^2=1$

$P=a$ $\quad Q = \sqrt{1-a^2}$ $\quad Z = aX + \sqrt{1-a^2}\,Y + b$

$$2\sqrt{z} = a\log x + \log y \cdot \sqrt{1-a^2} + c.$$

$$\log x^a + \log y^{\sqrt{1-a^2}} - \log c_1 = 2\sqrt{z}$$

$$\log\{x^a y^{\sqrt{1-a^2}}/c_1\} = 2\sqrt{z} \Rightarrow \boxed{x^a y^{\sqrt{1-a^2}} = c_1 e^{2\sqrt{z}}}$$

Exercise

Ex-1 Solve the following equations

(a) $x^2p^2 + y^2q^2 = z^2$

(b) $z^2 = pqxy$

Standard forms-II (only z, p, q present)

Rule-I To Solve $f(z, p, q) = 0$

Rule-II Take $p = aq$ in the given equation and Solve for p and q

Rule-III put in $dz = pdx + qdy$ and integrate.

Ex Find the complete integral of the equations

(a) $p^2z^2 + q^2 = 1$ (b) $9(p^2z + q^2) = 4$

(a) $\Rightarrow$ put $q = ap$

$$p^2z^2 + a^2p^2 = 1 \Rightarrow p = \frac{1}{\sqrt{z^2+a^2}}$$

$$dz = pdx + qdy$$

$$dz = \frac{1}{\sqrt{z^2+a^2}}dx + ady$$

$$\sqrt{z^2+a^2}\,dz = dx + ady.$$

integrating, we obtain

$$\frac{z}{2}\sqrt{z^2+a^2} + \frac{a^2}{2}\log(z+\sqrt{z^2+a^2}) = x + ay + b.$$

(b) $\Rightarrow$ put $q = ap$ $\quad 9(p^2z + a^2p^2) = 4$

$$p^2(z+a^2) = \frac{4}{9}.$$

$$p = \frac{2}{3\sqrt{z+a^2}} \Rightarrow q = \frac{2a}{3\sqrt{z+a^2}}$$

$$dz = pdx + qdy \Rightarrow dz = \frac{2}{3\sqrt{z+a^2}}dx + \frac{2a}{3\sqrt{z+a^2}}dy.$$

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$$\frac{3}{2}\sqrt{z+a^2}\,dz = dx + a\,dy$$

integrating $(z^2+ay)^{3/2} = x + ay + b$

$$(z+ay)^3 = (x+ay+b)^2$$

Ex Find the complete integral of the equations
(a) $p^2-q^2=z$, (b) $z^2(p^2x^2+q^2)=1$

Standard form – III → $f(x,p) = g(y,q)$

Rule
S-1 write the given equations in the form
$f(x,p) = g(y,q)$
S-2 Take $f(x,p) = g(y,q) = a$ (Const)
Solve these equations for p and q.
S-3 put in $dz = p\,dx + q\,dy$

Ex-1 Find the complete integral of the equations
$$x(1+y)p = y(1+x)q.$$

we have $\frac{xp}{1+x} \cancel{p} = \frac{yq}{1+y} = a$ (Const).

$$p = \frac{1+x}{x}a \qquad q = \frac{1+y}{y}a.$$

putting these values of p and q in $dz = p\,dx + q\,dy$

$$dz = \frac{(1+x)a}{x}dx + \frac{(1+y)a}{y}dy$$

integrating $z = (\log x + x)a + (\log y + y)a + b$

$$\boxed{z = a\log xy + ax + ay + b}$$

Ex Solve $p^2y(1+x^2) = qx^2$

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a^2 \text{ (Say)}$$

$$p = \frac{ax}{\sqrt{1+x^2}},\ q = a^2y \Rightarrow dz = \frac{ax}{\sqrt{1+x^2}}dx + a^2y\,dy$$

$$\boxed{z = a\sqrt{1+x^2} + \frac{1}{2}a^2y^2 + b}$$

Ex Solve $p^2q^2 + x^2y^2 = x^2q^2(x^2+y^2)$

we have $\frac{p^2}{x^2} + \frac{y^2}{q^2} = x^2+y^2$

$\Rightarrow \frac{p^2}{x^2} - x^2 = y^2 - \frac{y^2}{q^2} = a^2$ (say)

$\therefore p = x\sqrt{x^2+a^2}, \quad q = \frac{y}{\sqrt{y^2-a^2}}$.

The equation $dz = pdx + qdy$

$dz = x\sqrt{x^2+a^2}\,dx + \frac{y}{\sqrt{y^2-a^2}}\,dy$

Integrating $z = \frac{1}{3}(x^2+a^2)^{3/2} + \sqrt{y^2-a^2} + b$

Ex Solve $p^2q(x^2+y^2) = p^2+q$

$x^2+y^2 = \frac{1}{q} + \frac{1}{p^2}$

or $\frac{1}{p^2} - x^2 = y^2 - \frac{1}{q} = a^2$ (say)

$p = \frac{1}{\sqrt{x^2+a^2}} \quad q = \frac{1}{y^2-a^2}$.

The equation $dz = pdx + qdy$ becomes.

$dz = \frac{1}{\sqrt{x^2+a^2}}\,dx + \frac{1}{y^2-a^2}\,dy$.

Integrating $z = \log[x+\sqrt{x^2+a^2}] + \frac{1}{2a}\log\left|\frac{y-a}{y+a}\right| + b$.

Exercise

Find the complete integral of the equations

(a) $p^2+q^2 = z^2(x+y)$

(b) $z(p^2-q^2) = x-y$

(c) $zpy^2 = x(y^2+z^2q^2)$

(d) $p+q-2px-2qy+1=0$

Standard Form - IV (Clariaut form)

$$z = px + qy + f(p,q) \qquad — ①$$

$$f \equiv px + qy + f(p,q) - z$$

Charpit auxiliary equations $\qquad f_x = p,\ f_y = q,\ f_z = -1$

$$\frac{dp}{0} = \frac{dq}{0} \Rightarrow \boxed{p = a}\ \boxed{q = b}$$

Substituting these values in ① $\boxed{z = ax + by + f(a,b)}$

Ex Solve $\quad z = px + qy + c\sqrt{1+p^2+q^2}$

$$z = ax + by + c\sqrt{1+a^2+b^2}$$

Ex Solve $(p+q)(z-px-qy) = 1.$

Sol Re-writting the given equation in the standard form

$z = px + qy + f(p,q)$ we get.

$$z = px + qy + \frac{1}{p+q}$$

$$\boxed{z = ax + by + \frac{1}{a+b}}$$

Ex Solve $\quad 2zx - px^2 - 2qxy + pq = 0$

The given equation can be written in the form

$$2xz = px^2 + 2qxy - pq$$

$$z = \frac{1}{2}px + qy - \frac{pq}{2x}.$$

put $x^2 = u$, so that $p = P(2x)$, $P = \frac{\partial z}{\partial u}$.

Equation (2) can now be written as

$$z = Pu + qy - Pq$$

$$z = au + by - ab$$

$$\boxed{z = ax^2 + by - ab}$$

Classification of P.D.E.. Reduction to canonical or Normal forms. (Riemann Method)

⇒ Classification of P.D.E of 2nd order →

Consider a general p.d.e of second order for a function of two independent variables x and y in the form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{---} (1)$$

Where R, S, and T are continuous function of x and y only possessing partial differential ~~Equation~~ derivatives defined in some domain D on the xy plane. Then (1) is said to be

(i) Hyperbolic if $S^2 - 4RT > 0$

(ii) Parabolic if $S^2 - 4RT = 0$

(iii) Elliptic if $S^2 - 4RT < 0$

Examples (i) Consider the one-dimensional wave Equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} \quad \text{i.e} \quad r - t = 0$$

Comparing it with (1) here $R = 1,\ S = 0,\ T = -1$

Hence $S^2 - 4RT = 0 - 4 \times (1)(-1) = 4 > 0$

The given equation is hyperbolic

(ii) The one-dimensional diffusion equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} \quad \text{is parabolic} \ [S^2 - 4RT = 0 - 4.1.0 = 0$$

(iii) The two dimensional harmonic equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{is elliptic} \quad [S^2 - 4RT = 0 - 4.1.1 = -4 < 0$$

EX

Ex Classify the following P.d.E

(i) $2\left(\frac{\partial^2 u}{\partial x^2}\right) + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 2$

(ii) $u_{xx} + 4u_{xy} + 4u_{yy} = 0$

(iii) $xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$

Reduction to canonical Form of Hyperbolic Equations

Working Rule →

Step-1 A second order P.d.E

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{---- (1)}$$

is hyperbolic, if $S^2 - 4RT > 0$

Step-2 Solve the quadratic Equation

$$R\lambda^2 + S\lambda + T = 0 \quad \text{---- (2)}$$

Let λ_1 and λ_2 be its two distinct roots of (2)

Step-3 Then the corresponding, characteristic Equation are

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0$$

Solving ~~(1)~~ these, we get $f_1(x,y) = C_1$, and $f_2(x,y) = C_2$

Step-4 We select u, v such that $u = f_1(x,y)$ and $v = f_2(x,y)$

use these transformation to find $p, q, r, s,$ and t in terms of u and v.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2} \text{ etc.}$$

Step-5 on putting these values of, p, q, r, t, the given equation reduces to canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

Ex Reduce the equation $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form.

Sol Given $r^2 - x^2 t = 0$

comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we get

$$R = 1, \; S = 0, \; T = -x^2$$

$\therefore$ $S^2 - 4RT = 4x^2 > 0$ (hyperbolic type,) for $x \neq 0$

The λ-quadratic equation reduces to $R\lambda^2 + S\lambda + T = 0$

$\lambda^2 - x^2 = 0$

$\Rightarrow \lambda = \pm x$ (real and distinct roots

The corresponding characteristic Equation

$$\frac{dy}{dx} + \lambda_i(x,y) = 0 \Rightarrow \frac{dy}{dx} \pm x = 0$$

Integrating them $\quad y + \frac{1}{2}x^2 = c_1, \quad y - \frac{1}{2}x^2 = c_2$

Choose $\quad u = y + \frac{1}{2}x^2$, and $v = y - \frac{1}{2}x^2$ —(1)

$$\therefore p = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}x - \frac{\partial z}{\partial v}x \qquad \because \frac{\partial u}{\partial x} = x,\ \frac{\partial v}{\partial x} = -x$$

$$q = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad \frac{\partial u}{\partial y} = 1,\ \frac{\partial v}{\partial y} = 1$$

Now $\quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$= x^2\frac{\partial^2 z}{\partial u^2} - 2x^2\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \quad —(2)$$

Similarly,

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \quad —(3)$$

Putting (2) and (3) in the given equation we get

$$4x^2\frac{\partial^2 z}{\partial u\partial v} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}.$$

using (1) $\quad \frac{\partial^2 z}{\partial u\partial v} = \frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$ which is the required canonical form of the given equation

Exercise

Ex-1 Reduce the equation to canonical form $\quad y^2 r - x^2 t = 0$

Ex-2 Reduce the equation $\quad (n-1)^2\frac{\partial^2 z}{\partial x^2} - y^{2n}\frac{\partial^2 z}{\partial y^2} = ny^{2n-1}\frac{\partial z}{\partial y}$ to canonical form, and find its general solution.

Reduction to canonical form of elliptic Equations:-

Working Rule → S-1 Let the equation $Rr+Ss+Tt+f(x,y,z,p,q)=0$ —(1)

be elliptic so that $S^2-4RT<0$ —— (2)

Step-2 write α quadratic equation $R\alpha^2+S\alpha+T=0$ - - - (3)

Its two roots α_1 and α_2 are complex conjugates.

Step-3 Then corresponding characteristic equations are

$$\frac{dy}{dx}+\alpha_1=0 \quad \text{and} \quad \frac{dy}{dx}+\alpha_2=0$$

Solving these $f_1(x,y) \pm f_2(x,y) = \text{constant}$ $i=\sqrt{-1}$

Step-4 we take $u=f_1(x,y)+if_2(x,y)$, $v=f_1(x,y)-if_2(x,y)$

$$u=\alpha+i\beta \qquad v=\alpha-i\beta$$

$$\alpha=f_1(x,y) \text{ and } \beta=f_2(x,y)$$

Use these transformation to find

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x}. \quad q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial p}{\partial x}, \quad s=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial p}{\partial y}, \quad t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial q}{\partial y}$$

Step-5 On putting the values of p, q, r, s, t the given equation reduces to the canonical form

$$\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=F\left(\alpha,\beta,z,\frac{\partial z}{\partial \alpha},\frac{\partial z}{\partial \beta}\right)$$

Ex Reduce the equation $\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}=y$ to canonical form.

Sol Given $r+y^2t-y=0$, —(1) $S^2-4RT\alpha=-4y^2<0$ for $y\neq 0$

Elliptic

The α-quadratic Equation $R\alpha^2+S\alpha+T=0 \Rightarrow \alpha^2+y^2=0 \Rightarrow \alpha=iy,-iy$

The corresponding char Equ $\frac{dy}{dx}+iy=0$ $\frac{dy}{dx}-iy=0$

Integrating $\log y+ix=c_1$ and $\log y-ix=c_2$.

Choose $u=\log y+ix=\alpha+i\beta$, $v=\log y-ix=\alpha-i\beta$

$\alpha=\log y$, and $\beta=x$. —(2)

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x}=\frac{\partial z}{\partial \beta}.$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y}=\frac{1}{y}\frac{\partial z}{\partial \alpha}$$

The λ-quadratic equation reduces to $R\lambda^2+S\lambda+T=0$

$$\lambda^2-x^2=0$$

$$\Rightarrow \lambda=\pm x \quad \text{(real and distinct roots}$$

The corresponding characteristic Equation

$$\frac{dy}{dx}+\lambda_i(x,y)=0 \Rightarrow \frac{dy}{dx}\pm x=0$$

Integrating them $\quad y+\frac{1}{2}x^2=C_1, \quad y-\frac{1}{2}x^2=C_2$

Choose $\quad u=y+\frac{1}{2}x^2$, and $v=y-\frac{1}{2}x^2$ — (1)

$$\therefore p=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}x-\frac{\partial z}{\partial v}x \quad \left| \begin{array}{l} \frac{\partial u}{\partial x}=x, \ \frac{\partial v}{\partial x}=-x \\ \frac{\partial u}{\partial y}=1, \ \frac{\partial v}{\partial y}=1 \end{array}\right.$$

$$q=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$$

Now $\quad r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left[x\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right]$

$$=x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

$$=x^2\frac{\partial^2 z}{\partial u^2}-2x^2\frac{\partial^2 z}{\partial u\partial v}+x^2\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v} \quad \text{— (2)}$$

Similarly, $\quad \frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$ — (3)

Putting (2) and (3) in the given equation we get

$$4x^2\frac{\partial^2 z}{\partial u\partial v}=\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}.$$

using (1) $\quad \frac{\partial^2 z}{\partial u\partial v}=\frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$ which is the required canonical form of the given equation

Exercise

Ex-1 Reduce the equation to canonical form $y^2r-x^2t=0$

Ex-2 Reduce the equation $(n-1)^2\frac{\partial^2 z}{\partial x^2}-y^{2n}\frac{\partial^2 z}{\partial y^2}=ny^{2n-1}\frac{\partial z}{\partial y}$ to canonical form, and find its general solution.

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x} = \frac{\partial^2 z}{\partial \beta^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial z}{\partial \alpha}\right) = -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= \frac{1}{y^2}\left(\frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha}\right)$$

using r, t in ① the required canonical form is

$$\frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha} - y = 0,$$

$$\boxed{\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{\partial z}{\partial \alpha} + e^{\alpha}}$$

Exercise

Ex-1 Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + x^2\frac{\partial^2 z}{\partial y^2} = 0$$

to canonical form.

Ex-2 Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + 5\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} - 3z = 0$$

to canonical form.

Reduction to Canonical Form of Parabolic Equations →
(Parabolic form)

Working Rule →

Step-1 A second order partial differential Equation:

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

is parabolic if $S^2 - 4RT = 0$

Step-2 Solve the quadratic equation $R\lambda^2 + S\lambda + T = 0$, which gives two equal roots λ_1, λ_2 (say)

Step-3 Solve the first order ordinary differential equation

$$\frac{dy}{dx} + \lambda_1 = 0$$

Let its solution be $f(x, y) = c$.

Step-4 We take $u = f(x, y)$ and $v = g(x, y)$

Where $g(x, y)$ is an arbitrary function of x and y is independent of u

i.e $J = \frac{\partial(u, v)}{\partial(x, y)} = u_x v_y - u_y v_x \neq 0$

Step-5 On putting the values of p, q, r, s and t the given equation reduces to canonical form:

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

or $$\frac{\partial^2 z}{\partial u^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

Ex-1 Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

to Canonical form and hence solve it

Sol:- The given equation is $r + 2s + t = 0$

$\therefore$ $R = 1$, $S = 2$, $T = 1$ $\Rightarrow$ $S^2 - 4RT = 0$ (parabolic)

$R\lambda^2 + S\lambda + T = 0$, becomes $(\lambda + 1)^2 = 0$

$\therefore$ $\lambda = -1$ is the only distinct root.

The equation $\frac{dy}{dx} + \lambda(x, y) = 0 \Rightarrow \frac{dy}{dx} - 1 = 0$

On integrating $x - y = \text{constant}$

Choose as $u = x - y$, other independent variables $v = x + y$

So that $$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 1.1 + 1.1 = 2 \neq 0$$

We have $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial y} + \frac{\partial^2 z}{\partial v^2}.$$

$$s = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting these values of r, s, t in the given equation, we get

$\frac{\partial^2 z}{\partial v^2} = 0$ which is the canonical form

Integrating w.r. to v (Keeping u constant) we obtain

$$\frac{\partial z}{\partial v} = \phi(u)$$

Again $$z = v\phi(u) + \psi(u)$$

Hence $$z = (x+y)\,\phi(x-y) + \psi(x-y)$$

ϕ and ψ are arbitrary function

Exercise

Ex-1 Reduce $r + 2xs + x^2 t = 0$ to canonical form

Ex-2 Reduce $r - 2s + t + p - q = 0$ to canonical form and hence solve it.