Semester - VI
Subject : Numerical methods
Class : BSC (PS)/(CS)
Department : Mathematics

Numerical integration

The Process of evaluating a definite integral ~~form~~ from a set of tabulated values of the integrand $f(x)$, which is not known explicitly is called numerical integration. In this section we ~~derive~~ ~~to~~ study several formulae for numerical integration

Trapezoidal rule

Let $I = \int_a^b f(x)\,dx$, where $f(x)$ takes the values $y_0, y_1, ---, y_n$ for $x = x_0, x_1, ---, x_n$.

(2)

~~Let~~ Let us divide the interval (a,b) ~~[illegible]~~ into n subintervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $\ldots$, $x_n = x_0 + nh = b$.

$$I = \int_a^b f(x)\,dx = h\int_0^n f(x_0 + ph)\,dp$$

when $p = \frac{x - x_0}{h}$

$$= h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots\right] \times dp$$

(by newton forward formula)

$$= h\left[p y_0 + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 y_0 + \cdots\right]_0^n$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \cdots\right]$$

This is called Newton - Cote's quadrature formula.

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For $n=1$

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{h}{2}\left[y_0 + y_1\right]$$

Similarly,

$$\int_{x_1}^{x_1+h} f(x)\,dx = \frac{h}{2}\left[y_1 + y_2\right]$$

$$\vdots$$

$$\int_{x_{n-1}}^{x_n} f(x)\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

Adding all equations, we get

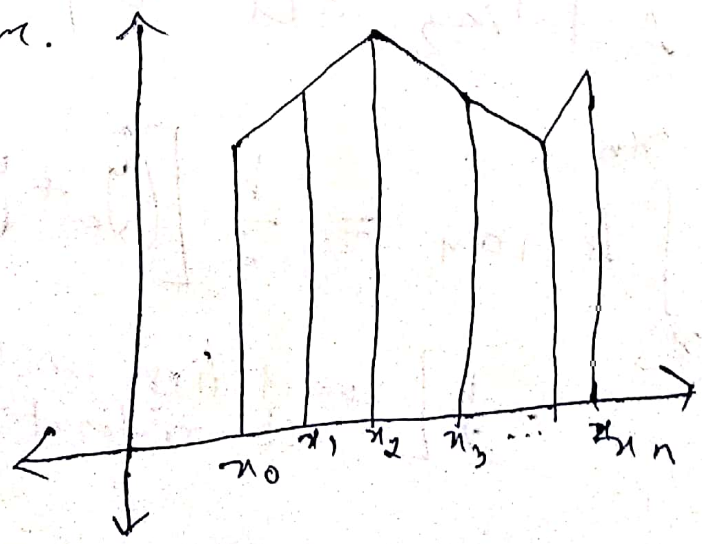
$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\right]$$

$$= \frac{h}{2}\left[\text{sum of first and last ordinate} + 2(\text{sum of intermediate ordinates})\right]$$

Geometrical interpretation of Trapezoidal rule :

The definite integral $\int_{x_0}^{x_1} f(x)\,dx$ represents the area bounded by the curve $y = f(x)$, the ordinate $x = x_0$, $x = x_n$ and the x-axis.

In trapezoidal rule the curve $y = f(x)$ is replaced by n straight line segments joining the points (x_0, y_0), (x_1, y_1), ---, (x_n, y_n) as shown in figure.



Area $\int_{x_0}^{x_n} f(x)\,dx$ is approximately given by the sum of area of the n trapeziums obtained. Hence, this method is known as trapezoidal rule.

Error in trapezoidal rule

$$|E| < \frac{(b-a)h^2}{12} M, \text{ where}$$

$$M = \max\{|y_0''|, |y_1''|, \ldots, |y_{n-1}''|\}$$

Problem 1.

Evaluate $\int_0^1 \frac{dx}{1+x^2}$, using Trapezoidal rule with $h = 0.2$. Hence determine the value of π.

Solution. Here $h = 0.2$.

Let $y = \frac{1}{1+x^2}$

$a = 0$, $b = 1$

$n = \frac{b-a}{h} = \frac{1-0}{0.2} = 5$

The values of y for each point of subdivision $h = 0.2$ are got and tabulated as.

	x_0	x_1	x_2	x_3	x_4	x_5
x	0	0.2	0.4	0.6	0.8	1
y	1	0.9615	0.8621	0.7353	0.6098	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

By Trapezoidal rule.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.2}{2}\Big[1 + 0.5 + 2\big(0.9615 + 0.8621$$
$$+ 0.7353 + 0.6098\big)\Big]$$

$$= 0.7837 \quad \text{—— (1)}$$

Actul value

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x\Big]_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

—— (2)

From (1) and (2),

$$\frac{\pi}{4} = 0.7837$$

$$\pi = 3.1348 \quad (\text{Appro.})$$