Linear Programing & Theory of Games.

Theorems/Results on Simplex Method & Duality

Theorem 1 – Improved Basic Feasible Solution.

Statement – Let $\bar{x}_B$ be a basic feasible solution to the L.P.P. : Maximize $Z = \bar{c}\bar{x}$ subject to $A\bar{x} = \bar{b}$; $\bar{x} \geq \bar{0}$.
Let $\hat{x}_B$ be another basic feasible solution obtained by admitting a non-basis column vector $\bar{a}_j$ in the basis, for which the net evaluation $z_j - c_j$ is negative. Then, $\hat{x}_B$ is an improved basic feasible solution to the problem, that is,

$$\hat{c}_B \hat{x}_B > \bar{c}_B \bar{x}_B.$$

Proof – The L.P.P. is to determine $\bar{x}$, so as to Maximize $z = \bar{c}\bar{x}$; $\bar{c}, \bar{x}^T \in \mathbb{R}^n$, subject to the constraints : $A\bar{x} = \bar{b}$ and $\bar{x} \geq \bar{0}$, $\bar{b}^T \in \mathbb{R}^m$, where A is an $m \times n$ real matrix.
We are given that, $\bar{x}_B$ is a basic feasible solution.

$$\text{Let } z_0 = \bar{c}_B \bar{x}_B.$$

Let $\hat{a}_j$ be the column vector introduced in the basis, such that $z_j - c_j < 0$.

Let $\bar{b}_r$ be the vector removed from the basis & let $\hat{x}_B$ be the new basic feasible solution.
Then,

$$\hat{x}_{Bi} = x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}} \quad \& \quad \hat{x}_{Br} = \frac{x_{Br}}{y_{rj}}$$

Thus, the new value of the objective function is,

$$\hat{z} = \sum_{i=1}^{m} \hat{c}_{Bi} \cdot \hat{x}_{Bi}$$

$$= \sum_{i=1}^{m} \bar{c}_{Bi}\left(\bar{x}_{Bi} - \bar{x}_{Br} \cdot \frac{y_{ij}}{y_{rj}}\right) + \hat{c}_{Br} \frac{x_{Br}}{y_{rj}}$$

$$= \sum_{i=1}^{m} \bar{c}_{Bi}\left(\bar{x}_{Bi} - \bar{x}_{Br} \frac{y_{ij}}{y_{rj}}\right) + c_j \cdot \frac{x_{Br}}{y_{rj}}$$

$\{$Since $\hat{c}_{Br} = c_j\}$

$$= z_0 - (z_j - c_j) \cdot \frac{x_{Br}}{y_{rj}}$$

$> z_0$ $\{$ Since, $x_{Br}/y_{rj} > 0 \}$

Hence, the new basic feasible solution $\hat{x}_B$ gives an improved value of the objective function.

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Theorem 2– If a linear programming problem, Max. $Z = \bar{c}\bar{x}$

Such that, $A\bar{x} = \bar{b}$, $\bar{x} \geq \bar{0}$, has an optimal feasible solution, then atleast one basic feasible solution must be optimal.

Proof– Let $Z_0 = \bar{c}_B \cdot \bar{x}_B$ with $\bar{x}_B = B^{-1} \cdot \bar{b}$, be a basic feasible solution to the problem.

Now, to show that, The basic feasible solution is optimal, we are required to prove $Z_0 \geq Z^*$ where Z^* is the optimal value, given by,

$$Z^* = \sum_{j=1}^{N} c_j \cdot x_j .$$

The constraint equation $A\bar{x} = \bar{b}$ can be expressed as,

$$\sum_{j=1}^{N} a_j \cdot x_j = \bar{b} \qquad \text{——— (i)}$$

Since, any vector $a_j \in A$ can be expressed as a linear combination of vectors in B, i.e.

$$a_j = \sum_{i=1}^{m} y_{ij} \beta_j \qquad \text{——— (ii)}$$

Substituting (ii) in (i); We get,

$$\sum_{j=1}^{N} \left(\sum_{i=1}^{m} y_{ij} \cdot \beta_j \right) x_j = \bar{b}.$$

or,

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{N} y_{ij} \cdot x_j \right) \beta_j = \bar{b}$$

Since, $\bar{x}_B = B^{-1} \bar{b}$ is the basic feasible solution.

$$\sum_{i=1}^{m} x_{Bi} \cdot \beta_i = \bar{b}$$

Any vector can be uniquely expressed as a function of its basis vectors,

Therefore,

$$\bar{x}_B = \sum_{i=1}^{m} x_{Bi} \cdot \beta_i$$

The optimality condition is

$$z_j - c_j \geq 0 \quad \& \quad x_j \geq 0 .$$

Therefore, $$\sum_{j=1}^{N} x_j \cdot z_j \geq \sum_{j=1}^{N} c_j \cdot x_j = z^* \quad \text{——— (iii)}$$

But, we have,

$$\sum_{j=1}^{N} x_j \cdot z_j \geq \sum_{j=1}^{N} x_i \left(\sum_{i=1}^{m} c_{Bi} \cdot y_{ij} \right)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{N} x_i \cdot y_{ij} \right) c_{Bi}$$

$$= \sum_{i=1}^{m} x_{Bi} \cdot c_{Bi} = z_0 \quad \text{——— (iv)}$$

So, From (iii) & (iv),

$$z_0 \geq z^*$$

Thus, The basic feasible solution is optimal.

Theorem 3 – If X is any feasible solution to the primal problem & W is any feasible solution to the dual problem, then $\bar{c}\bar{x} \le \bar{b}^T\bar{w}$ i.e. $z_x \le z_w$

Alternative Statement – Given problems are;

$$A\bar{x} \le \bar{b}, \ \bar{x} \ge \bar{0} \ ; \ \text{Max. } z_x = \bar{c}\bar{x} \quad \text{---(i)}$$

$$A^T\bar{w} \ge \bar{c}^T, \ \bar{w} \ge \bar{0} \ ; \ \text{Min } z_w = \bar{b}^T\bar{w} \quad \text{---(ii)}$$

Then, for any feasible solution $\bar{x}$ to (i) and any feasible solution $\bar{w}$ to (ii), $\bar{c}\bar{x} \le \bar{b}^T\bar{w}$ i.e. $z_x \le z_w$.

Proof – Let $\bar{x}$ & $\bar{w}$ be any feasible solutions to the primal and the dual problems respectively.

We have,

$$\bar{c}\bar{x} = \sum_{j=1}^{n} c_j \cdot x_j \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \cdot w_i \right) x_j$$

$$= \sum_{i=1}^{m} w_i \left(\sum_{j=1}^{n} a_{ij} \cdot x_j \right) \le \sum_{i=1}^{m} w_i \cdot b_i = \bar{b}^T\bar{w}$$

So, $\bar{c}\bar{x} \le \bar{b}^T\bar{w}$ or $z_x \le z_w$.

This completes the proof of the theorem.

Theorem 6·6. *If the kth constraint of the primal is an equality, then the dual variable w_k is unrestricted in sign.* **[Meerut 90]**

Proof. As pointed out earlier, if the kth constraint is an equality, then we can write the primal in the form:

$$\left.\begin{array}{l} c_1x_1 + c_2x_2 + \ldots + c_nx_n = z \text{ (max.)} \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n \le b_k \\ -a_{k1}x_1 - a_{k2}x_2 - \ldots - a_{kn}x_n \le -b_k \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m, \end{array}\right\} \quad \ldots(6.22)$$

so that its dual is given by

$$\left.\begin{array}{l} b_1w_1 + b_2w_2 + \ldots + b_k\,(w_k' - w_k'') + \ldots + b_m w_m = z_w \text{ (min.)} \\ a_{11}w_1 + a_{21}w_2 + \ldots + a_{k1}\,(w_k' - w_k'') + \ldots + a_{m1}w_m \ge c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{k2}\,(w_k' - w_k'') + \ldots + a_{m2}w_m \ge c_2 \\ \vdots \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{kn}\,(w_k' - w_k'') + \ldots + a_{mn}w_m \ge c_n. \end{array}\right\} \quad \ldots(6{\cdot}23)$$

If we replace $(w_k' - w_k'')$ by a new variable w_k, then we observe that the system (6·23) becomes equivalent to (6·2) except that w_k is unrestricted in sign.

Theorem 6·7. *If the pth variable of the primal is unrestricted in sign, the pth constraint of the dual is an equality.*

Proof. Let the given primal be of the form :

$$\text{Max. } z_x = c_1x_1 + c_2x_2 + \ldots + c_px_p + \ldots + c_nx_n,$$

$$\begin{aligned} \text{subject to } \quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}x_p + \ldots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2p}x_p + \ldots + a_{2n}x_n &\le b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mp}x_p + \ldots + a_{mn}x_n &\le b_m \end{aligned}$$

where $x_1, x_2, \ldots, x_{p+1}, \ldots, x_n \ge 0$, but x_p is unrestricted in sign.

Since the variable x_p (unrestricted in sign) can always be expressed as the difference of two non-negative variables, say x_p' and x_p'', therefore x_p can be replaced by $(x_p' - x_p'')$ in the above primal problem.

Thus, we have the new primal of the form :

$$\text{Max. } z_x = c_1x_1 + c_2x_2 + \ldots + c_p(x_p' - x_p'') + \ldots + c_nx_n, \text{ subject to}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}(x_p' - x_p'') + \ldots + a_{1n}x_n &\le b_1 \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mp}(x_p' - x_p'') + \ldots + a_{mn}x_n &\le b_m, \end{aligned}$$

where

$$x_1, x_2, \ldots, x_p', x_p'', \ldots, x_n \ge 0.$$

It's dual is given by, Min. $z_w = b_1w_1 + b_2w_2 + \ldots + b_mw_m$, subject to

$$\begin{aligned} a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m &\ge c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m &\ge c_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ \left\{ \begin{aligned} a_{1p}w_1 + a_{2p}w_2 + \ldots + a_{mp}w_m &\ge c_p \\ -a_{1p}w_1 - a_{2p}w_2 - \ldots - a_{mp}w_m &\ge -c_p \end{aligned} \right\} \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m &\ge c_n \end{aligned}$$

6·5-3 Complementary Slackness Theorem

In this section, we shall make a further effort to relate the primal and dual formulations of linear programming problem. The following theorem (called the ***complementary slackness theorem***) shows the intimate relation between primal and the dual.

Theorem 6·11. ***(Complementary Slackness Theorem)*** *For the optimal feasible solutions of the primal and dual systems,*

(i) if the inequality occurs in the ith relation of either system (the corresponding slack or surplus variable x_{n+i} is + ive), then the ith variable w_i of its dual is zero, i.e. $w_i = 0$;

(ii) if the jth variable x_j is + ive in either system, the jth dual constraint holds as a strict equality (i.e. the corresponding slack or surplus variable w_{m+j} is zero).

Proof. Since the primal objective function in explicit form can be written as

$$c_1x_1 + c_2x_2 + \dots + c_nx_n = z_x \text{ (max.)} \qquad \dots(6\cdot24)$$

and the dual objective function can be written as

$$b_1w_1 + b_2w_2 + \dots + b_mw_m = z_w \text{ (min.)} \qquad \dots(6\cdot25)$$

Also, the primal constraint-inequalities can be wirtten as equations after introducing the non-negative slack variables $x_{n+1}, x_{n+2}, \dots, x_{n+m}$ as follows :

$$\left.\begin{array}{lll} a_{11}x_1 + \dots + a_{1n}x_n + x_{n+1} & & = b_1 \\ \vdots \qquad \vdots \qquad \vdots & & \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n & + x_{n+m} & = b_m . \end{array}\right] \qquad \dots(6\cdot26)$$

Similarly, the dual-inequalities can be written as equations after introducing the non-negative surplus variables $w_{m+1}, w_{m+2}, \dots, w_{m+n}$ as follows :

$$\left.\begin{array}{lll} a_{11}w_1 + \dots + a_{m1}w_m - w_{m+1} & & = c_1 \\ \vdots \qquad \vdots \qquad \vdots & & \vdots \\ a_{1n}w_1 + \dots + a_{mn}w_m & - w_{m+n} & = c_n \end{array}\right] \qquad \dots(6\cdot27)$$

Now, multiplying the equations in (6·26) by $w_1, w_2, \dots, w_m$, respectively, and adding the resultant set, we get

$$x_1 \sum_{i=1}^{m} a_{i1}w_i + x_2 \sum_{i=1}^{m} a_{i2}w_i + \dots + x_n \sum_{i=1}^{m} a_{in}w_i + x_{n+1}w_1 + x_{n+2}w_2 + \dots + x_{n+m}w_m = \sum_{i=1}^{m} w_ib_i \quad \dots(6\cdot28)$$

Next, we subtract (6·28) from (6·24) and obtain

$$(c_1 - \sum_{i=1}^{m} a_{i1}w_i)\, x_1 + (c_2 - \sum_{i=1}^{m} a_{i2}w_i)\, x_2 + \dots + (c_n - \sum_{i=1}^{m} a_{in}w_i)\, x_n - w_1x_{n+1} - w_2x_{n+2} - \dots - w_mx_{n+m}$$

$$= \left(z_x - \sum_{i=1}^{m} w_i b_i \right) \qquad \text{...(6·29)}$$

But, from (6·25), we have

$$z_w = \sum_{i=1}^{m} w_i b_i \qquad \text{...(6·30)}$$

and from (6·27), we have

$$-w_{m+j} = c_j - (a_{1j} w_1 + a_{2j} w_2 + \ldots + a_{mj} w_m) \text{ or } -w_{m+j} = c_j - \sum_{i=1}^{m} a_{ij} w_i , j = 1, 2, \ldots, n \qquad \text{...(6·31)}$$

Now, using the results (6·30) and (6·31), eqn. (6·29) becomes

$$-(w_{m+1} x_1 + w_{m+2} x_2 + \ldots + w_{m+n} x_n) - (w_1 x_{n+1} + w_2 x_{n+2} + \ldots \ldots + w_m x_{n+m}) = z_x - z_w . \qquad \text{...(6·32)}$$

If $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ and $\mathbf{w}^* = (w_1^*, w_2^*, \ldots, w_m^*)$ be the optimum solutions to the primal and dual problems respectively, then

$$z_x^* = z_w^* \qquad \text{...(6·33)}$$

as claimed by the *Duality Theorem*.

Thus, for these optimum solutions and corresponding slack surplus variables

$$x_{n+i}^* \geq 0 \ (i = 1, 2, \ldots, m) \text{ and } w_{m+j}^* \geq 0 \ (j = 1, 2, \ldots, n),$$

the equation (6·32) becomes

$$-(w_{m+1}^* x_1^* + w_{m+2}^* x_2^* + \ldots + w_{m+n}^* x_n^*) - (w_1^* x_{n+1}^* + \ldots + w_m^* x_{n+m}^*) = 0 \text{ [from (6·33)]} \qquad \text{...(6·34)}$$

Since all the variables in (6·34) are restricted to be non-negative, all the product terms of (6·34) are also non-negative, and for the validity of (6·34), each term must individually be equal to zero.

Thus, $w_{m+j}^* x_j^* = 0$, and $w_i^* x_{n+i}^* = 0$, for all i and j.

(a) Now, if $w_{m+j}^* > 0$, we must have $x_j^* = 0$; if $x_{n+i}^* > 0$, then $w_i^* = 0$, which proves the first part of the theorem.

(b) If $x_j^* > 0$, then $w_{m+j}^* = 0$, *i.e.*, the jth constraint of dual problem is strict equality ; if $w_i^* > 0$, then $x_{n+i}^* = 0$, *i.e.*, the ith primal constraint is strict equality, which proves the second part of the theorem.

Hence the theorem is now completely established.