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B.Sc (H) - sem - II, electricity & magnetism

Prob - 1 :- Derive the following relations with the help of the maxwell's equations

(i) $F = \frac{1}{4\pi\epsilon_0} \frac{q q_0}{r^2}$ (coulomb's law in electrostatics)

(ii) $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$ (the equation of continuity)

(iii) $\vec{\nabla}\cdot\vec{B} = 0$ (maxwell's equation)

(iv) $\vec{\nabla}\cdot\vec{D} = \rho$ (maxwell's equation)

Ans (i) : We know that

$$\oint_S \vec{A}\cdot d\vec{s} = \oint_V (\vec{\nabla}\cdot\vec{A})\, dV \rightarrow (1)$$ [Gauss's divergence theorem]

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \rightarrow (2)$$ (maxwell's equation)

Now, taking volume integrating of eq (2)

$$\oint_V (\vec{\nabla}\cdot\vec{E})\, dV = \oint_V \left(\frac{\rho}{\epsilon_0}\right) dV$$ using eq (1)

$$\oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0}\oint_V \rho\, dV$$

if the electric field around the charge is symmetrical, then

$$E(4\pi r^2) = \frac{1}{\epsilon_0}\times(q)$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \rightarrow (3)$$

where $q \Rightarrow$ Total charge within the closed surface

if F is the force on a test charge q_0, then we know that

$$E = F/q_0$$

$$\Rightarrow F = Eq_0 \text{ using eq } (3)$$

$$\boxed{F = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}}$$

this is the Coulomb's law in electrostatics

(ii) from maxwell's fourth equation

$$\text{curl } \vec{B} = \mu_0\left(J + \frac{\partial \vec{D}}{\partial t}\right)$$

$$\text{or } \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \rightarrow (1)$$

Taking the divergence on both sides of eq (1)

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot\mu_0\left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

$$0 = \mu_0\left[\vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})\right] \rightarrow (2)$$

$\because$ Since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$

by maxwell's first equation

$$\vec{\nabla}\cdot\vec{D} = \rho \rightarrow (3)$$

from (2) & (3)

$$\boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0}$$ proved

(iii) From third maxwell's equation

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

taking the divergence on both sides of above equation

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B})$$

$$0 = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B})$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$ Hence proved.

(iv) from fourth maxwell's equation

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \rightarrow (1)$$

taking divergence on both sides of eq (1)

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot\mu_0\left[\vec{J}+\frac{\partial\vec{D}}{\partial t}\right]$$

$$0 = \mu_0\left[\vec{\nabla}\cdot\vec{J}+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})\right]$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J}+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) = 0 \rightarrow (2)$$

We know that the equation of continuity

$$\boxed{\vec{\nabla}\cdot\vec{J}+\frac{\partial \rho}{\partial t} = 0 \rightarrow (3)}$$

from (2) & (3)

$$-\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{D} = \rho} \text{ or } \boxed{\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0}$$

Hence Proved $\quad (\because \vec{D} = \epsilon_0\vec{E})$

Prob 2

find the electromagnetic wave equations in vacuum or free space and expression for velocity of the wave equation

Ans! We know that, the maxwell's equation are

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \longrightarrow (1)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \longrightarrow (2)$$

$$\vec{\nabla}\times\vec{E} = \cancel{\rho/\epsilon_0} -\frac{\partial B}{\partial t} \longrightarrow (3)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right) \rightarrow (4)$$

for free space $\rho = 0$, $\vec{J} = 0$

$$\vec{\nabla}\cdot\vec{E} = 0 \rightarrow (5)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \rightarrow (6)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow (7)$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} \rightarrow (8)$$

Now taking curl on both sides of equation (7)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$(\vec{\nabla}\cdot\vec{E})\vec{\nabla} - (\vec{\nabla}\cdot\vec{\nabla})\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$ using eq (5) & (8)

$$0 - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left[\mu_0 \frac{\partial \vec{D}}{\partial t}\right]$$

$$\nabla^2\vec{E} = \mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}$$ $\because \vec{D} = \epsilon_0 \vec{E}$

$$\nabla^2\vec{E} = \mu_0 \frac{\partial^2}{\partial t^2}(\epsilon_0 \vec{E})$$

$$\boxed{\nabla^2\vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \rightarrow (9)$$

this is the wave equation for the propagation of electric field $\vec{E}$ in vacuum or free space

Similarly, taking curl on both sides of eq (8)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left(\mu_0 \frac{\partial \vec{D}}{\partial t}\right)$$

$$(\vec{\nabla} \cdot \vec{B})\vec{\nabla} - (\vec{\nabla} \cdot \vec{\nabla})\vec{B} = \mu_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{D})$$

$$(\because \vec{D} = \epsilon_0 \vec{E})$$

$$(\vec{\nabla} \cdot \vec{B})\vec{\nabla} - (\vec{\nabla} \cdot \vec{\nabla})\vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$ using equation (6) & (7)

$$0 - \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left[-\frac{\partial \vec{B}}{\partial t}\right]$$

$$\boxed{\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}} \rightarrow (10) \quad (\because \vec{B} = \mu_0 \vec{H})$$

or

$$\boxed{\nabla^2 \vec{H} = \mu_0 \epsilon_0 \frac{\partial^2 H}{\partial t^2}}$$

this is the wave equation for the propagation of magnetic field $\vec{B}$, in vacuum or free space

We know that general equation of wave propagation with a speed v at any time t is given by

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \rightarrow (11)$$

Comparing the equating (9) & (10) with eq (11)

then $\frac{1}{v^2} = \mu_0 \epsilon_0$

$$\boxed{v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}} \rightarrow (12)$$

this is expression for velocity of electromagnetic waves in vacuum

Prob (3) A coil is placed in a magnetic field $\vec{B}$ where the direction of magnetic field is normal to the plane of the coil. The magnetic flux linked with the coil depends on the time t as follows

$$\phi = (3t^2 - 6t + 5) \text{ weber}$$

Calculate the induced e.m.f. in the coil at t = 2 sec.

Ans :- Given that $\phi = (3t^2 - 6t + 5)$ weber & $t = 2$ sec

But we know that, induced e.m.f.

$$e = -\frac{d\phi}{dt}$$

$$e = -\frac{d}{dt}(3t^2 - 6t + 5)$$

$$e = -(6t - 6 + 5)$$

at t = 2 sec, $e = -(6 \times 2 - 6 + 5)$

$$\boxed{e = -11 \text{ Volt}}$$ Ans

Prob-4 :- A coil of area of cross section 100 cm^2, having 50 turns of wire is placed in a magnetic field of intensity 2×10^{-2} T with its plane normal to the field. If the coil is withdrawn from the field in $\frac{1}{10}$ second, calculate the induced e.m.f. in the coil.

Solution

Given that

$$a = 100\,cm^2 = 0.01\,m^2$$

$$n = 50,\ B = 2\times10^{-2}\,T$$

Initial condition, magnetic flux linked with coil

$$\phi_1 = Bna$$

$$\phi_1 = 2\times10^{-2}\times 50\times 0.01 = 10^{-2}\,Wb$$

and magnetic flux linked with coil after

$$t = \frac{1}{10}\,sec,\ \phi_2 = 0$$

Now

$$\text{induced e.m.f.} = -\frac{d\phi}{dt}$$

$$e = -\frac{(\phi_2-\phi_1)}{t}$$

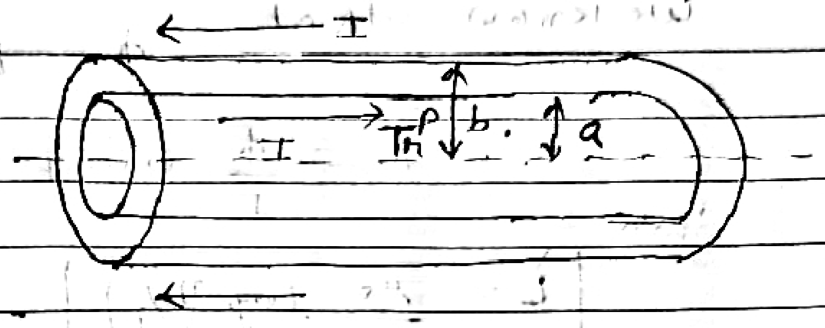
$$e = -\frac{(0-10^{-2})}{\left(\frac{1}{10}\right)} = 0.1\ \text{Volt}$$

Prob-5

A long coaxial cable has radius of its outer cylinder b and of inner cylinder a, if a current I is taken by the outer cylinder and is brought back by the inner cylinder of the cable as shown in fig, then calculate

(a) Self Inductance of the cable per unit length

(b) the magnetic energy stored in length l of the cable.



Solution :- (a) Let P be any point from axis of cable. then we know that, the magnetic field at point P is given by

$$B = \frac{\mu_0 I}{2\pi r} \rightarrow (1)$$

Now consider a small element of unit length between the radii r and r + dr, normal to the $\vec{B}$ then the magnetic flux linked with this elementary area da is

where l is the length of the cable

$$d\phi = B\,da = \frac{\mu_0 I}{2\pi r}(l\,dr)$$

∴ Total magnetic flux linked between the cylinders

$$\phi = \int_a^b \frac{\mu_0 I\, l\,dr}{2\pi r} = \frac{\mu_0 I l}{2\pi}\int_a^b \frac{1}{r}\,dr$$

$$= \frac{\mu_0 I l}{2\pi}\left(\log_e r\right)_a^b$$

$$= \frac{\mu_0 I l}{2\pi}\left\{\log_e b - \log_e a\right\}$$

$$\phi = \frac{\mu_0 I l}{2\pi}\log_e\left(b/a\right) \rightarrow (2)$$

(10)

We know that

$$\phi = LI$$

$$\therefore L = \frac{\phi}{I} \quad \text{using equation (2)}$$

$$L = \frac{\mu_0 l}{2\pi} \log_e (b/a) \rightarrow (3)$$

then

$$\boxed{L = \frac{\mu_0}{2\pi} \log_e (b/a)} \rightarrow (4)$$

at $l = 1$

this is self inductance of the cable per unit length

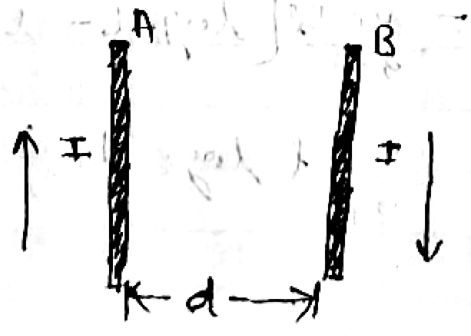
(b) We know that, the magnetic energy stored in the cable

$$U = \frac{1}{2} L I^2 \quad \text{using equation (3)}$$

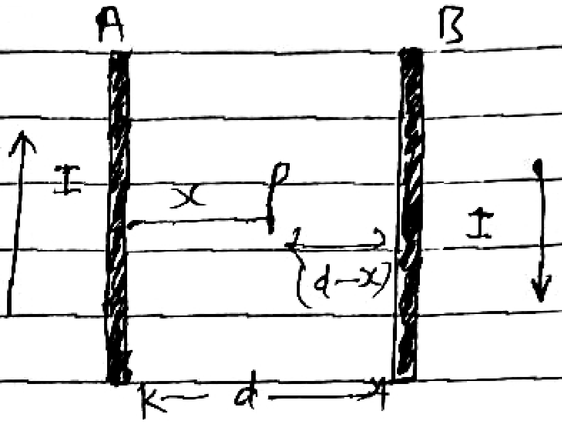
$$U = \frac{1}{2} \left[\frac{\mu_0 l}{2\pi} \log_e (b/a)\right] I^2$$

$$\boxed{U = \frac{\mu_0 I^2 l}{4\pi} \log_e (b/a)} \rightarrow (4)$$

Prob 6 Consider two parallel wires each of length l and radius r and separation d, as show in fig. If a current I is taken by a wire and is brought back by the other wire, calculate the self inductance of the arrangement.



Solution:-



Let P be any point from wire A where we have to calculate magnetic field

Now, the magnetic field at point P due to wire A

$$B_1 = \frac{\mu_0 I}{2\pi x} \otimes \text{(inward)} \rightarrow (1)$$

Similarly, magnetic field at point P due to wire B

$$B_2 = \frac{\mu_0 I}{2\pi (d-x)} \otimes \text{ inward}$$

therefore, Total magnetic field at point P is given by

$$B = B_1 + B_2 = \frac{\mu_0 I}{2\pi}\left(\frac{1}{x} + \frac{1}{d-x}\right) \otimes \rightarrow (3)$$

Now consider a small elementary area of length l & thickness dx at the point P. then the magnetic flux ~~linked with~~ is given by

$$d\phi = B\,da = \frac{\mu_0 I}{2\pi}\left(\frac{1}{x} + \frac{1}{d-x}\right)(l\,dx)$$

∴ Total magnetic flux between the wires

$$\phi = \int_{r}^{(d-r)} \frac{\mu_0 I}{2\pi}\left(\frac{1}{x} + \frac{1}{d-x}\right) l\,dx$$

$$\phi = \frac{\mu_0 I l}{\pi} \log_e\left(\frac{d-r}{r}\right)$$

or

$$\boxed{\frac{\phi}{I} = L = \frac{\mu_0 l}{\pi} \log_e\left(\frac{d-r}{r}\right)}$$