Bivariate Normal Distribution :

- It is the generalisation of one variable Normal Distribution
- Bivariate ⟶ Two Random variables
  ↓
  joint pdf, mean, variance, co-rellation coefficient

Definition : A pair of random variables X and Y have a bivariate normal distribution and they are refferd as Jointly Distributed Normal variables iff their Joint Probability Density Function is given by

$$f(x,y) = \frac{e^{-\frac{1}{2(1-\rho)^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$, where $\sigma_1 > 0$, $\sigma_2 > 0$. and $\boxed{-1 < \rho < 1}$.

σ_1^2 = Varaince of X , σ_2^2 = Variance of Y , ρ = Corellation Coeffcients

Following Question Could be Asked .

1. Marginal Density Funcition ⟶ $f_X(x)$ and $f_Y(y)$
2. Mean μ_1 , μ_2
3. Variance σ_1^2 and σ_2^2.
4. Co-variance ($Cov(x,y)$) & Co-Relation Cofficient $(\rho) = \frac{Cov(x,y)}{\sigma_1 \sigma_2}$.
5. **THEOREM** : If X and Y have a bivariate normal distribution, the conditional, probability density function (pdf) [$f_{Y|X}$] of Y given X = x is a Normal Distribution with mean (Conditional Mean)

$$\boxed{\mu_{Y|X} = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x-\mu_1)}$$

& Variance $\sigma^2_{Y|X} = \sigma_2^2 (1-\rho^2)$ ⇒ Conditional Variance.

Similarly $\mu_{X|Y} = \bigcirc$

& $\sigma^2_{X|Y} = \square$

6. $\boxed{\text{THEOREM}}$: If two Random variables have bivariate normal distribution, they are independent iff $\boxed{\rho = 0}$

$\boxed{\text{HINTS}}$:

1. Marginal Density Functions

$$f_X(x) = \underbrace{\int_{-\infty}^{\infty} f(x,y)\, dy}_{\text{Bivariate Normal distribution}} = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)^2}.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\left(\frac{y-\mu_2}{\sigma_2}\right)^2}$$

2. $E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \mu_1$, $E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \mu_2$. (Easy to show)

3. $\sigma_1^2 = E(X^2) - \mu_1^2$, $\sigma_2^2 = E(Y^2) - \mu_2^2$ | Variances.

4. $\mathrm{COV}(X,Y) = E(XY) - \mu_1 \cdot \mu_2$, $\rho = \frac{\mathrm{COV}(X,Y)}{\sigma_1 \sigma_2}$ (Co-Relation Coefficient)

5. $\boxed{\text{THEOREM}}$:

Find $f_{Y|X} = \frac{f(x,y)}{f_X(x)} = \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left[\frac{y - \{\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\}}{\sigma_2\sqrt{1-\rho^2}}\right]^2}$

joint normal pdf

Use Formula to PROVE IT.

Conditional Probability Density function

marginal pdf for x.

$\longrightarrow$ Conclude Mean and Variance.

6. $\boxed{\text{THEOREM}}$: Use the Previous Chapter Result ↓

Theorem 2.5.1. *Let the random variables X_1 and X_2 have supports S_1 and S_2, respectively, and have the joint pdf $f(x_1, x_2)$. Then X_1 and X_2 are independent if and only if $f(x_1, x_2)$ can be written as a product of a nonnegative function of x_1 and a nonnegative function of x_2. That is,*

$$f(x_1, x_2) \equiv g(x_1)h(x_2),$$

where $g(x_1) > 0$, $x_1 \in S_1$, zero elsewhere, and $h(x_2) > 0$, $x_2 \in S_2$, zero elsewhere.

Then

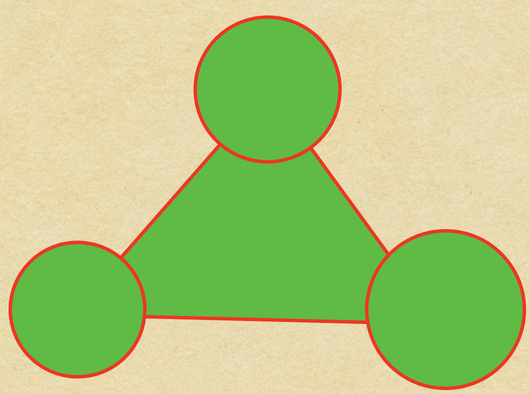
$$f(x,y) = \frac{e^{-\frac{1}{2(1-\rho)^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

(Suppose X & Y are independent)

$= f_X(x) \cdot f_Y(y)$ | marginal pdf of X & Y.

$$= \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \cdot \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}$$

$\Longleftrightarrow$ $\boxed{\rho = 0}$ Check this (Easy)



Exercise: Bivariate Normal Distribution

Q1. To prove Theorem at 5 S.N, show that if X and Y have a bivariate normal distribution, then
(a) their independence implies that $\rho = 0$;
(b) $\rho = 0$ implies that they are 49.

Q2. If X and Y have a bivariate normal distribution, it can be shown that their joint moment-generating function is given by

$$\begin{aligned} M_{X,Y}(t_1, t_2) &= E(e^{t_1 X + t_2 Y}) \\ &= e^{t_1\mu_1 + t_2\mu_2 + \frac{1}{2}(\sigma_1^2 t_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2)} \end{aligned}$$

Verify that
(a) the first partial derivative of this function with respect to
(b) the second partial derivative with respect to t_1 at $t_1 = 0$ and $t_2 = 0$ is $\sigma_1^2 + \mu_1^2$.
(c) the second partial derivative with respect to t_1 and t_2 at $t_1 = 0$ and $t_2 = 0$ is $\rho\sigma_1\sigma_2 + \mu_1\mu_2$.

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