Q4 For any $b \in \mathbb{R}$, prove that $\lim\left(\frac{b}{n}\right) = 0$

Ans:-

Let $\varepsilon > 0$ be given,

$\Rightarrow \frac{1}{\varepsilon} > 0$.

Let $b > 0$ [otherwise take $-b > 0$]

$\left\{ \text{For } b = 0 \quad \left(\frac{b}{n}\right) = (0, 0, \dots) \to 0 \right\}$

$\therefore \frac{b}{\varepsilon} > 0$.

$\left|\frac{b}{n} - 0\right| = \frac{b}{n}$ — (1)

For $\frac{b}{\varepsilon} > 0$

By Archimedean property, $\exists$ a $K \in \mathbb{N}$ s.t.

$\frac{b}{\varepsilon} < K \Rightarrow \frac{b}{K} < \varepsilon$. — (2)

Now For $n \geq K$.

$\Rightarrow \frac{1}{n} \leq \frac{1}{K}$.

$\Rightarrow \frac{b}{n} \leq \frac{b}{K}$ $[b > 0]$

$\therefore$ by (1) & (2).

$$\left|\frac{b}{n} - 0\right| = \frac{b}{n} \leq \frac{b}{K} < \varepsilon.$$

$\therefore$ for every $\varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\left|\frac{b}{n} - 0\right| < \varepsilon \ \forall n \geq K$.

$\Rightarrow \lim\left(\frac{b}{n}\right) = 0$

2nd method.

$\lim\left(\frac{b}{n}\right) = b \cdot \lim\left(\frac{1}{n}\right) = b \cdot 0 = 0$ [by algebra of limits]

[Prove it here]

Q5 Use the definition of limit of a sequence to establish the following limits.

(d) $\lim \left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$.

Ans:- Let $\varepsilon > 0$ be given.

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{2n^2-2-2n^2-3}{4n^2+6}\right|$$

$$= \left|\frac{-5}{4n^2+6}\right|$$

$$= \frac{5}{4n^2+6}$$

$$< \frac{5}{n^2} < \frac{5}{n} \quad \text{--- (1)}$$

[$4n^2+6 > n^2$
$\Rightarrow \frac{1}{4n^2+6} < \frac{1}{n^2}$

& $n^2 > n$.
$\frac{5}{n^2} < \frac{5}{n}$.

For $\frac{5}{\varepsilon} > 0$
by Archimedean property,
$\exists\, K \in \mathbb{N}$ s.t. $\frac{5}{\varepsilon} < K \Rightarrow \frac{5}{K} < \varepsilon$ --- (2)

Now for $n \geq K$

$\frac{1}{n} \leq K \Rightarrow \frac{5}{n} \leq \frac{5}{K}$

by (1) & (2)

$$\left|\frac{n^2-1}{2n^2-3} - \frac{1}{2}\right| < \frac{5}{n} \leq \frac{5}{5} < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary,
$\therefore$ for every $\varepsilon > 0$, $\exists\, K \in \mathbb{N}$ s.t.

$$\left|\frac{n^2-1}{2n^2-3} - \frac{1}{2}\right| < \varepsilon \quad \forall\, n \geq K$$

$$\Rightarrow \lim \left(\frac{n^2-1}{2n^2-3}\right) = \frac{1}{2}.$$

Q 6 Show that

(a) $\lim \left(\frac{1}{\sqrt{n+7}}\right) = 0$

Ans: Let $\varepsilon > 0$ be given.

Now $\left|\frac{1}{\sqrt{n+7}} - 0\right| = \frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}}$ — (1)

$\left[\because n+7 > n \quad \sqrt{n+7} > \sqrt{n} \Rightarrow \frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}}\right]$

$\varepsilon > 0$
$\Rightarrow \frac{1}{\varepsilon^2} > 0.$

$\therefore$ by Archimedean property, $\exists K \in \mathbb{N}$ s.t.
$\frac{1}{\varepsilon^2} < K.$
$\Rightarrow \frac{1}{\varepsilon} < \sqrt{K}.$
$\Rightarrow \varepsilon > \frac{1}{\sqrt{K}}$ — (2)

For $n \geq K \Rightarrow \sqrt{n} \geq \sqrt{K} \Rightarrow \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{K}}.$

by (1) & (2)

$$\left|\frac{1}{\sqrt{n+7}} - 0\right| < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{K}} < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary,

$\therefore$ for every $\varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\left|\frac{1}{\sqrt{n+7}} - 0\right| < \varepsilon$ $\forall n \geq K.$

$\therefore \lim \left(\frac{1}{\sqrt{n+7}}\right) = 0$

Q7 Let $x_n = \frac{1}{\ln(n+1)}$ for $n \in \mathbb{N}$.

(a) Use the definition of limit to show that $\lim (x_n) = 0$

(b) Find a specific value of $K(\varepsilon)$ as required in the definition of limit for each of

(i) $\varepsilon = \frac{1}{2}$.

(ii) $\varepsilon = \frac{1}{10}$.

Q9 Prove that $\lim(x_n)=0$ iff $\lim(|x_n|)=0$. (3)
Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).

Ans:- Let $\lim(x_n)=0$. $\Rightarrow$ for every $\varepsilon>0$ $\exists$ $K\in\mathbb{N}$ s.t. $|x_n-0|<\varepsilon$ $\forall n\geq K$ — (1)
Let $\varepsilon>0$ be given
(1) $\Rightarrow$ $\exists$ $K\in\mathbb{N}$ s.t. $\big||x_n|-0\big|<\varepsilon$ $\forall n\geq K$
For $n\geq K$.
Now $\big||x_n|-0\big| = \big||x_n|\big| = |x_n| = |x_n-0| < \varepsilon$ [by (1)]

$\therefore$ $\lim(|x_n|)=0$

$\Leftarrow$ Let $\lim(|x_n|)=0$
$\Rightarrow$ for every $\varepsilon>0$ $\exists$ $K\in\mathbb{N}$ s.t. $\big||x_n|-0\big|<\varepsilon$. $\forall n\geq K$ — (2)

For $n\geq K$.
$|x_n-0| = |x_n| = \big||x_n|\big| = \big||x_n|-0\big| < \varepsilon$ by (2)
$\Rightarrow$ $\lim(x_n)=0$

Example

Let $x_n=(-1)^n$; $n\in\mathbb{N}$
$(|x_n|) = (1,1,1,\dots)$
$(x_n) = (-1,1,-1,1,\dots)$
we know $(|x_n|)\to 1$ (being constant seq)
but (x_n) does not converge [Prove it yourself done in class]
$\therefore$ $\boxed{(|x_n|) \text{ converges} \not\Rightarrow (x_n) \text{ converges}}$

Q10 Prove that if $\lim(x_n)=x$ and if $x>0$, then $\exists$ a $M\in\mathbb{N}$ s.t. $x_n>0$ $\forall n\geq M$.

Ans:- Let $\lim(x_n)=x$, $x>0$
Let $\varepsilon<x$ $\Rightarrow$ $x-\varepsilon>0$.
For ~~x~~ $\varepsilon>0$ $\exists$ $M\in\mathbb{N}$ s.t. $|x_n-x|<\varepsilon$ $\forall n\geq M$

$\therefore$ For $n \geq M$

$|x_n - x| < \varepsilon$

$\Rightarrow -\varepsilon < x_n - x < \varepsilon$

$\Rightarrow x - \varepsilon < x_n < x + \varepsilon$

$\Rightarrow 0 < x - \varepsilon < x_n < x + \varepsilon$
$\downarrow$
+ve

$\therefore$ $\boxed{x_n > 0 \ \forall \ n \geq M}$

Q 11 Show that $\lim \left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$

Ans:- Let $\varepsilon > 0$ be given

$$\left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{n+1-n}{n(n+1)}\right|$$

$$= \left|\frac{1}{n(n+1)}\right|$$

$$= \frac{1}{n(n+1)} < \frac{1}{n}. \quad -(1)$$

Now

$n^2 + n > n.$

$\frac{1}{n^2+n} < \frac{1}{n}$

For $\varepsilon > 0$.

by Archimedean property, $\exists K \in \mathbb{N}$ s.t

$\frac{1}{\varepsilon} < K \Rightarrow \varepsilon > \frac{1}{K}.$

For $n \geq K \Rightarrow \frac{1}{n} \leq \frac{1}{K}. \quad -(2)$

by (1) & (2)

$$\left|\left[\frac{1}{n} - \frac{1}{n+1}\right] - 0\right| < \frac{1}{n} \leq \frac{1}{K} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary

$\therefore$ for every $\varepsilon > 0$ $\exists K \in \mathbb{N}$ s.t. $\left|\frac{1}{n} - \frac{1}{n+1} - 0\right| < \varepsilon$

$\forall n \geq K.$

$\Rightarrow \lim \left(\frac{1}{n} - \frac{1}{n+1}\right) = 0.$

Q 16

Show $\lim \left(\frac{2^n}{n!} \right) = 0$

Ans:- Let $x_n = \frac{2^n}{n!}$

x_n is positive

$\Rightarrow (x_n)$ is a positive term seq.

Now, $\frac{x_{n+1}}{x_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1} = \frac{2/n}{1+1/n}$

we know $\lim\limits_{n \to \infty} \left(\frac{2/n}{1+1/n} \right) = 0 < 1$

$\therefore$ by thm 3.2.11 (state)

$\lim \left(\frac{2^n}{n!} \right) = 0.$

Q13 Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim (nb^n) = 0$

Ans:- As $0 < b < 1$

Let $b = \frac{1}{1+a}$ f.s $a > 0$.

Now $(1+a)^n = 1 + na + \frac{1}{2} n(n-1) a^2 + \cdots > \frac{1}{2} n(n-1) a^2$,

$\therefore \frac{1}{(1+a)^n} < \frac{2}{n(n-1)a^2}$

$|nb^n - 0| = \left| \frac{n}{(1+a)^n} \right| = \frac{n}{(1+a)^n} < \frac{2n}{n(n-1)a^2}$

$= \frac{2}{(n-1)a^2}$

$= \left(\frac{2}{a^2} \right) \cdot \frac{1}{(n-1)}$

Let $c = \frac{2}{k^2}$.

$\therefore |nb^n - 0| < c\frac{1}{n-1}$. for $n>1$.

Since $\lim\left(\frac{1}{n-1}\right) = 0$. where $\left(\frac{1}{n-1}\right)$ is +ve term seq

$\therefore$ by thm 3.1.10. (state).

$$\boxed{\lim(nb^n) = 0}$$

Q14 Show that $\lim\left((2n)^{1/n}\right) = 1$.

Ans:- Note $(2n)^{1/n} > 1 \ \forall n \in \mathbb{N}$.

Let $(2n)^{\frac{1}{n}} = 1 + k_n$ s.t. $k_n > 0$. — (A)

$\Rightarrow \quad 2n = (1+k_n)^n = 1 + nk_n + \frac{1}{2}n(n-1)k_n^2 + \cdots$

$\geq 1 + \frac{1}{2}n(n-1)k_n^2$.

$\Rightarrow \quad (2n-1) \geq \frac{1}{2}n(n-1)k_n^2$.

$\Rightarrow \quad k_n^2 \leq \frac{4n-2}{n(n-1)} = \frac{2}{n} + \frac{2}{n-1} \leq \frac{2}{n-1} + \frac{2}{n-1}$

$= \frac{4}{n-1}$ — (1)

$n > n-1$.

$\frac{1}{n} < \frac{1}{n-1}$.

Let $\varepsilon > 0$ be given. $\quad k_n \leq \frac{2}{\sqrt{n-1}}$

$\Rightarrow \frac{4}{\varepsilon^2} + 1 > 0$.

by Archimedean property $\exists K \in \mathbb{N}$ s.t.

$\frac{4+\varepsilon^2}{\varepsilon^2} = \frac{4}{\varepsilon^2} + 1 < K. \quad \Rightarrow \frac{1}{K} < \frac{\varepsilon^2}{4+\varepsilon^2}$ — (2)

for $n \geq K \Rightarrow \frac{1}{n} \leq \frac{1}{K} \quad \Rightarrow \frac{1}{n} < \frac{\varepsilon^2}{4+\varepsilon^2} \Rightarrow \frac{2}{\sqrt{n-1}} < \varepsilon$ (3)

by (1) + (2) & (3)

$|(2n)^{1/n} - 1| = k_n$ by (A).

$\leq \frac{2}{\sqrt{n-1}} < \varepsilon$.

$\because \frac{2}{\sqrt{n-1}} < \varepsilon$

$\sqrt{n-1} > \frac{2}{\varepsilon}$

$n-1 > \frac{4}{\varepsilon^2}$

$\frac{1}{n} < \frac{\varepsilon^2}{4+\varepsilon^2} \Leftarrow n > \frac{4+\varepsilon^2}{\varepsilon^2} \Leftarrow n > \frac{4}{\varepsilon^2} + 1 \Rightarrow \frac{\varepsilon^2}{4} + 1 \Leftarrow n-1 > \frac{\varepsilon^2}{4}$

Since $\varepsilon > 0$ is arb.

$\therefore$ for every $\varepsilon > 0$ $\exists$ $k \in \mathbb{N}$ s.t

$$|(2n)^{1/n} - 1| < \varepsilon \quad \forall\, n \geq k$$

$$\Rightarrow \lim (2n)^{1/n} = 1.$$

Q17 If $\lim(x_n) = x > 0$, show that $\exists$ a $k \in \mathbb{N}$ s.t if $n \geq k$, $\frac{1}{2}x < x_n < 2x$.

Ans:- $\lim(x_n) = x > 0 \Rightarrow$

for every $\varepsilon > 0$, $\exists$ $k \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$

$$\Rightarrow x - \varepsilon < x_n < x + \varepsilon \quad \text{①}$$

In particular

For $\varepsilon = x$

$$\text{①} \Rightarrow \quad 0 < x_n < 2x. \quad -\text{②}$$

For $\varepsilon = \frac{x}{2}$.

$$\frac{x}{2} < x_n < \frac{3x}{2} \quad \cancel{< 2x} \quad \text{③}$$

Also $\frac{3x}{2} < 2x$. $[2x > 0]$

$\Rightarrow$ by ② & ③.

$$\boxed{\frac{x}{2} < x_n < 2x.}$$

H.P.

$x - \varepsilon = \frac{1}{2}x$

$x + \varepsilon = 2x$

$\varepsilon = x - \frac{1}{2}x = \frac{x}{2}$

$\varepsilon = 2x - x = x$.

Ex 3.2

Q1 For x_n given by the following formulas, establish either the convergence or the divergence of the sequence $X = (x_n)$

(a) $x_n = \dfrac{n}{n+1}$

$x_1 = \frac{1}{2}$, $x_2 = \frac{2}{3}$, ...

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \longrightarrow 1$

$0.5, 0.66, \ldots \qquad \frac{n}{n+1} < 1$

$\lim(x_n) = \lim\left(\frac{n}{n+1}\right) = \lim\left(\frac{\cancel{n}1}{1/n+1}\right)$

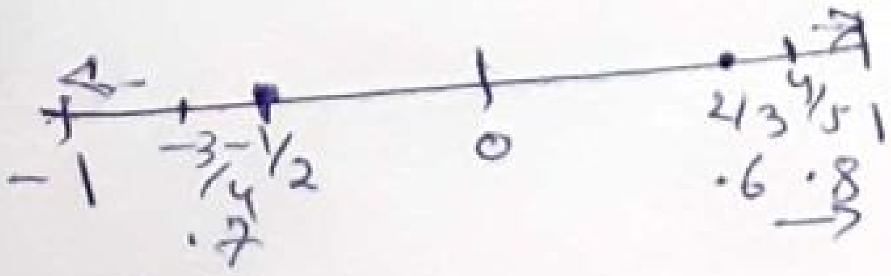
$= \dfrac{\lim(1)}{\lim\left(\frac{1}{n}\right) + \lim(1)}$ [by algebra of Limits]

$= \dfrac{1}{0+1}$

$= 1.$

(b) $x_n = \dfrac{(-1)^n n}{n+1}$

$x_1 = \frac{-1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$



Let $X' = \left(-\frac{1}{2}, -\frac{3}{4}, -\frac{5}{6}, \ldots\right)$

$= \left(\frac{-n}{n+1}\right)$ where n is odd.

Let $X'' = \left(\frac{2}{3}, \frac{4}{5}, \ldots\right) = \left(\frac{n}{n+1}\right)$ where n is even

Now $\lim\left(\frac{-n}{n+1}\right) = \lim\left(\frac{-1}{1+1/n}\right) = \dfrac{\lim(-1)}{\lim(1) + \lim\left(\frac{1}{n}\right)}$

$= \frac{-1}{1} = -1$

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and

$$\lim(x'') = \lim\left(\frac{n}{n+1}\right) = \frac{\lim(1)}{\lim(1)+\lim\left(\frac{1}{n}\right)} = 1.$$

$\therefore\ x' \to -1$

$x'' \to 1$

By divergence criteria,

$X = (x_n)$ is divergent.

Q2 Give an example of two divergent sequences X and Y s.t.

(a) their sum $X+Y$ converges

(b) their product XY converges

Ans:- Let $X = ((-1)^n)$

$Y = ((-1)^{n+1})$; $n \in \mathbb{N}$.

$X+Y = ((-1)^n + (-1)^{n+1})$; $n \in \mathbb{N}$

$= (0)$; $n \in \mathbb{N}$.

which converges to 0.

$XY = ((-1)^n \cdot (-1)^{n+1})$; $n \in \mathbb{N}$

$= \{(-1)\}$; $n \in \mathbb{N}$.

which converges to -1.

Q4 Show that if X and Y are sequences such that X converges to $x \neq 0$ and XY converges, then Y converges.

Ans:- XY converges.

also $X \to x \neq 0$ and clearly $X = (x_n)$ has non-zero real numbers.

$\therefore$ by thm 3.2.3 sequence $Y = \frac{XY}{X}$ converges.

Q5 Show that the following sequence are not convergent

(a) (2^n). (b) $((-1)^n n^2)$

Ans:- Let $X = (x_n) = (2^n)$.

Let X be convergent

$\Rightarrow$ X is bounded

$\Rightarrow \exists M \in \mathbb{R}$ s.t $|x_n| < M \quad \forall n \in \mathbb{N}$

$\Rightarrow |2^n| < M$

$\Rightarrow -M < 2^n < M$.

taking Log

$\Rightarrow n \log 2 < \log M$.

$\Rightarrow n < \frac{\log M}{\log 2}. \quad \forall n \in \mathbb{N}$

which is a contradiction to Archimedean property.

$\therefore$ X is not convergent

Q.6 Find the limit of following sequence.

(a) $\lim \left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)$ (b) $\lim \left(\frac{(-1)^n}{n+2}\right)$.

Ans:- (a) $\frac{\sqrt{n}-1}{(\sqrt{n}+1)} \frac{(\cancel{\sqrt{n}-1})}{(\cancel{\sqrt{n}-1})} = \frac{\cancel{n-1}}{\cancel{n}} \cdot \frac{1-1/\sqrt{n}}{1+1/\sqrt{n}}$.

$\lim \left(\frac{1-1/\sqrt{n}}{1+1/\sqrt{n}}\right) = \frac{\lim(1) - \lim(\frac{1}{\sqrt{n}})}{\lim 1 + \lim(1/\sqrt{n})}$

$= 1$

(b) $\lim \left(\frac{(-1)^n}{n+2}\right) = \lim \left(\frac{\frac{(-1)^n}{n}}{1+2/n}\right) = \frac{\lim \left(\frac{(-1)^n}{n}\right)}{\lim(1) + \lim\left(\frac{2}{n}\right)}$

$= 0$.

Q7 If (b_n) is a bounded sequence and $\lim(a_n)=0$, show that $\lim(a_n b_n)=0$. Explain why Thm 3.2.3 cannot be used.

Ans:-

(b_n) is bounded.

$\Rightarrow \exists\, M \in \mathbb{R}$ s.t $|b_n| \leq M \quad \forall n \in \mathbb{N}$.

$\lim(a_n)=0$

$\Rightarrow$ for every $\varepsilon_1 > 0$ $\exists$ $K \in \mathbb{N}$ s.t $|a_n - 0| < \varepsilon_1$ $\forall n \geq K$.

Now for $n \geq K$ and $\varepsilon_1 = \frac{\varepsilon}{M}$

$$|a_n b_n| = |a_n||b_n|$$
$$< M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

$\therefore$ $\lim(a_n b_n)=0$.

Thm 3.2.3 can not be used because.
(b_n) ~~is not~~ ~~con~~ may or may not be convergent

Q9 Let $y_n = \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that (y_n) and $(\sqrt{n}\, y_n)$ converges. Find their limits.

Ans:-

$$y_n = \sqrt{n+1} - \sqrt{n}.$$
$$= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} - \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$
$$= \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}}$$

$\left[\because \sqrt{n+1}+\sqrt{n} > \sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}} \right]$

Now $0 \leq y_n \leq \frac{1}{\sqrt{n}} \quad \forall n \in \mathbb{N}$.

Note $\lim(0) = 0 = \lim\left(\frac{1}{\sqrt{n}}\right)$.

$\therefore$ by squeeze thm $\lim(y_n) = 0$.

$$\sqrt{n}\, y_n = \sqrt{n}\left[\sqrt{n+1}-\sqrt{n}\right]$$
$$= \sqrt{n}\sqrt{n+1} - n.$$
$$= \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}$$
$$= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}$$
$$= \frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$
$$\therefore \lim_{n\to\infty}(\sqrt{n}\, y_n) = \lim\left(\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right)$$
$$= \frac{1}{2}.$$

Q10 Determine the following limits.

(a) $\lim \left((3\sqrt{n})^{1/2n}\right)$.

$$(3\sqrt{n})^{\frac{1}{2}\cdot\frac{1}{n}} = (\sqrt{3})^{1/n}\sqrt{n}^{1/n}.$$
$$\lim\left[(3\sqrt{n})^{1/2n}\right] = \lim\left((\sqrt{3})^{1/n}\right)\lim\left(\sqrt{n}^{1/n}\right).$$
$$= 1\cdot 1$$
$$= 1$$

$\left[\because \lim (c^{1/n}) = 1\right.$ if $c>0$

& $\lim(n^{1/n}) = 1$

$\left.\text{Prove these}\right]$

Q11 If $0<a<b$, determine
$$\lim\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right).$$

Ans:
$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{\left(\frac{a}{b}\right)^n\cdot a + b}{\left(\frac{a}{b}\right)^n+1}.$$

$\boxed{\text{as } \frac{a}{b}<1 \quad \therefore \lim\left(\frac{a}{b}\right)^n = 0}$

$$\therefore \lim\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right) = \frac{\lim\left(\frac{a}{b}\right)^n\cdot a + \lim b}{\lim\left(\frac{a}{b}\right)^n + \lim(1)} = b.$$

Q13 Use the squeeze thm to determine limit.

(a) $(n!)^{1/n^2}$

$$n! = 1\cdot 2\cdot 3\cdot 4 \cdots n \leq \underbrace{n\cdot n \cdots n}_{n \text{ times}}$$

$$n! \leq n^n.$$

$$\therefore (n!)^{1/n^2} \leq (n^n)^{1/n^2} = n^{1/n}.$$

Now

$$1 \leq (n!)^{1/n^2} \leq n^{1/n}.$$

we know $\lim(1) = \lim(n^{1/n}) = 1$.

$\therefore$ by squeeze thm $\lim (n!)^{1/n^2} = 1$.

Q14 Show that if $z_n = (a^n+b^n)^{1/n}$ where $0<a<b$, then $\lim(z_n) = b$.

$$z_n = b\cdot\left(\left(\frac{a}{b}\right)^n + 1\right)^{1/n}.$$

$$\leq b\cdot 2^{1/n}$$

as $a<b$

$\left[\Rightarrow \frac{a}{b} < 1 \quad \left(\frac{a}{b}\right)^n < 1\right]$

$\therefore \left(\left(\frac{a}{b}\right)^n + 1\right) < 2$

$\Rightarrow \left(\left(\frac{a}{b}\right)^n + 1\right)^{1/n} < 2^{1/n}$

$\therefore$

$$b \leq z_n \leq b\cdot 2^{1/n}. \quad \forall n \in \mathbb{N}.$$

Now $\lim b = \lim b\cdot 2^{1/n}$

$= b.$

$\left[\lim b \cdot \lim 2^{1/n} = b\cdot 1\right.$

$\left[\because \cancel{\lim} \; c>0, \; \lim c^{1/n} = 1\right]$

$\therefore$ by squeeze thm $\lim(z_n) = b.$

Q15 Find limit where a, b satisfy $0 < a < 1, b > 1$.

(i). $\left(\frac{n}{b^n}\right)$ (ii) $\left(\frac{2^{3n}}{3^{2n}}\right)$.

Ans:-

(i) Let $x_n = \frac{n}{b^n}$.

Note $\frac{n}{b^n}$ is the real number $\forall n$.

$$\frac{x_{n+1}}{x_n} = \frac{n+1/b^{n+1}}{n/b^n} = \left(\frac{n+1}{n}\right) \cdot \frac{1}{b}.$$

$$\lim \left(\frac{x_{n+1}}{x_n}\right) = \lim \left(\frac{1+1/n}{1}\right) \cdot \frac{1}{b}$$

$$= \frac{1}{b} < 1 \quad [\because b > 1 \Rightarrow \frac{1}{b} < 1]$$

$\therefore$ by thm 3.2.11,

$$\lim \left(\frac{n}{b^n}\right) = 0.$$

(ii) Let $x_n = \frac{2^{3n}}{3^{2n}}$.

Note $x_n \geq 0 \ \forall n$.

$$\frac{x_{n+1}}{x_n} = \frac{2^{3(n+1)}/3^{2(n+1)}}{2^{3n}/3^{2n}}.$$

$$= 2^3/3^2$$

$$= \frac{8}{9}.$$

$$\lim \left(\frac{x_{n+1}}{x_n}\right) = \frac{8}{9} < 1$$

$\therefore$ by thm 3.2.11, $\lim \left(\frac{2^{3n}}{3^{2n}}\right) = 0$

Q16 (a) Give an example of a convergent sequence (x_n) of positive number with $\lim\left(\frac{x_{n+1}}{x_n}\right)=1$

(b) Give ex. of a divergent seq with this property

Ans:-

(a) Let $(x_n)=(1)$.

then $\lim\left(\frac{x_{n+1}}{x_n}\right)=\lim\left(\frac{1}{1}\right)=1$.

and (x_n) is cgt

(b) Let $(x_n)=(n)$.

then $\lim\left(\frac{x_{n+1}}{x_n}\right)=\lim\left(\frac{n+1}{n}\right)=1$.

but (x_n) is divergent. (Prove it).

Q18 Discuss the converges where $0<a<1$, $b>1$

(i) (n^2a^n) (ii) $(n!/n^n)$

Q17 Let $X=(x_n)$ be a seq of +ve real numbers s.t $\lim\left(\frac{x_{n+1}}{x_n}\right)=L>1$. Show that X is not a bdd seq hence not cgt.

Ans 18:- Let $x_n=n^2a^n$.

Note $x_n\geq 0\ \forall n$

$$\frac{x_{n+1}}{x_n}=\frac{(n+1)^2a^{n+1}}{n^2a^n}=\left(\frac{n+1}{n}\right)^2\cdot a.$$

$$\lim\left(\frac{x_{n+1}}{x_n}\right)=\lim\left(\frac{n+1}{n}\right)\cdot\lim\left(\frac{n+1}{n}\right)\cdot\lim a$$

$$=a<1$$

$\therefore$ by thm 3.2.11 $\lim x_n=0$

(ii) Let $x_n = \frac{n!}{n^n}$.

Note $x_n \geq 0 \; \forall n$.

$$\frac{x_{n+1}}{x_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{n!/n^n} = \cancel{(n+1)^2 / \left(\frac{n+1}{n}\right)^n}.$$

$$= \cancel{\frac{(n+1)^2 \cdot (n+1)^n}{n^n}} \qquad = \frac{n+1}{(n+1)} \frac{n^n}{(n+1)^n}.$$

$$\cancel{= (n+1)^2 \left(1+\frac{1}{n}\right)^n}. \qquad = \left(\frac{n}{n+1}\right)^n.$$

$$= \left(\frac{1}{1+1/n}\right)^n.$$

$$\lim \left(\frac{x_{n+1}}{x_n}\right) = 1.$$

$$x_n = \frac{1}{n} \cdot \frac{2}{n} \cdots \left(\frac{n-1}{n}\right) \frac{n}{n}.$$

$$\therefore \lim x_n = \lim \frac{1}{n} \cdot \lim \frac{2}{n} \cdots \lim\left(\frac{n-1}{n}\right) \cdot \lim 1$$

$$= 0. \quad \text{"} \quad \text{"}$$

$$= 0.$$

$\therefore \boxed{\lim x_n = 0}$

Q19 Let (x_n) be a sequence of positive real numbers s.t $\lim (x_n^{1/n}) = L < 1$. Show that there exists a number r with $0 < r < 1$ s.t $0 < x_n < r^n$ for all sufficiently Large $n \in \mathbb{N}$. use this to show $\lim (x_n) = 0$

Ans:- As $\lim (x_n^{1/n}) = L$

$\therefore$ for every $\varepsilon > 0 \; \exists \, K \in \mathbb{N}$ s.t

$$|x_n^{1/n} - L| < \varepsilon \quad \forall n \geq k$$

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$\Rightarrow L-\varepsilon < x_n^{1/n} < L+\varepsilon$. — ① for $n \geq k$

Let r be a number s.t $L < r < 1$

Let $\varepsilon = r - L > 0$.

$\therefore$ ① $\Rightarrow$

$$0 < x_n^{1/n} < r$$

$$\Rightarrow 0 < x_n < r^n \quad \text{for } n \geq k.$$

Since $0 < r < 1$ $\quad \lim r^n = 0$

also $\lim(0) = 0$

$\therefore$ by squeeze thm

$$\lim (x_n) = 0.$$

Q20

(a) Give an example of a convergent seq (x_n) of the number with $\lim(x_n^{1/n}) = 1$.

(b) Give an example of a divergent seq (x_n) of the number with $\lim(x_n^{1/n}) = 1$.

Ans:-

(a) Let $(x_n) = (1)$

$$\lim (x_n^{1/n}) = 1$$

$$\& \ (x_n) \to 1$$

(b) Let $(x_n) = (n)$.

$$\lim (n^{1/n}) = 1$$

but (n) is divergent

Q21 Suppose that (x_n) is a convergent seq and (y_n) is such that for any $\varepsilon > 0$ $\exists$ M s.t. $|x_n - y_n| < \varepsilon$ $\forall n \geq M$. Does it follow that (y_n) is convergent?

Ans:- Let $\varepsilon > 0$ be given

given $(x_n) \to x$ (say)

$\therefore$ for $\frac{\varepsilon}{2} > 0$, $\exists$ $M \in \mathbb{N}$ s.t

$|x_n - x| < \frac{\varepsilon}{2}$ $\forall n \geq M$.

$\Rightarrow$ $|x_n - y_n + y_n - x| < \varepsilon$ for $n \geq M$.

Also given for any $\varepsilon > 0$

$|x_n - y_n| < \varepsilon$ $\forall n \geq M$

In particular for $\frac{\varepsilon}{2}$ true.

Now $|y_n - x|$

$= |y_n - x_n + x_n - x|$

$\leq |x_n - y_n| + |x_n - x|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$

As $\varepsilon > 0$ is arb.

$\therefore$ for every $\varepsilon > 0$, $\exists$ $M \in \mathbb{N}$ s.t $|y_n - x| < \varepsilon$ $\forall n \geq M$

$\therefore$ $\boxed{\lim (y_n) = x}$ i.e (y_n) is cgt.

Q22 Show that if (x_n) and (y_n) are cgt seq. then the seq (u_n) & (v_n) defined by $u_n = \max\{x_n, y_n\}$ & $v_n = \min\{x_n, y_n\}$ are also cgt.

Ans:- we know $u_n = \max\{x_n, y_n\}$

$= \frac{1}{2}(x_n + y_n + |x_n - y_n|)$

by algebra of Limits (Do it yourself)

Assignment 1

To be submitted by 4/5/2020

Q1 Use the definition of the limit of a sequence to establish the following limits.

(i) $\lim\left(\frac{2n}{n+1}\right)=2$ (ii) $\lim\left(\frac{3n+1}{2n+5}\right)=3/2$

Q2 Show that:

(i) $\lim\left(\frac{2n}{n+2}\right)=2$ (ii) $\lim\left(\frac{\sqrt{n}}{n+1}\right)=0$

(iii) $\lim\left(\frac{(-1)^n n}{n^2+1}\right)=0$

Q3 Let $x_n=\frac{1}{\ln(x+1)}$ for $n\in\mathbb{N}$.

(a) Use the definition of limit to show that $\lim(x_n)=0$

(b) Find a specific value of k as required in the definition of limit for each of

(i) $\varepsilon=1/2$ (ii) $\varepsilon=1/10$.

Q4 Show that if $x_n\geq 0$ $\forall n\in\mathbb{N}$ & $\lim(x_n)=0$ then $\lim(\sqrt{x_n})=0$.

Q5 Show that $\lim\left(\frac{1}{3^n}\right)=0$.

Q6 Show that $\lim(n^2/n!)=0$.

Q7 For x_n given by the following formulas, establish either the convergence or the divergence of the sequence $X=(x_n)$.

(i) $x_n=\left\{\frac{n^2}{n+1}\right.$ (ii) $x_n=\frac{2n^2+3}{n^2+1}$

Q8 Show that if X and Y are sequences s.t. X and X+Y are convergent, then Y is convergent.

Q9 Show that $((-1)^n n^2)$ is divergent

Q10 find the limit of the following seq.

(i) $\lim \left((2 + 1/n)^2 \right)$ (ii) $\lim \left(\frac{n+1}{n\sqrt{n}} \right)$.

Q11 Determinants $\lim \left((n+1)^{1/\ln(n+1)} \right)$.

Q12 If $a>0$, $b>0$ show $\left(\sqrt{(n+a)(n+b)} - n \right) = \frac{a+b}{2}$.

Q13 Determine Limit (use squeeze thm)

(i) (n^{1/n^2})

Q14 Determine Limit where a, b satisfy $0<a<1$, $b>1$

(i) (a^n) (ii) (n/b^n)